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Lesson 2. Quantum Bits, Gates, and Circuits

1. Opening and quantum circuit

A quantum circuit is a model used for quantum computation. It represents a sequence of quantum gate operations performed on quantum bits. You will learn quantum circuit model from analogies to classical computer logic circuits.

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2. Quantum Bits, Gates, and Circuits

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Qiskit

Qiskit Advocate, APAC Workforce and Education, IBM Quantum

Recent Activity:

- Quantum Tokyo: Local quantum community in Japan since 2020
- NICT Quantum Camp – Qiskit workshop since 2020
- Kawasaki Quantum Summer Camp since 2022
- Lectures at several universities

- MEXT Award 2024 in the field of science and technology

MEXT: the Ministry of Education, Culture, Sports, Science and Technology

The University of Tokyo
Special Lectures in Information Science II
Introduction to Near-Term Quantum Computing
情報科学科特別講義Ⅱ/量子計算論入門
2024年度の計画

Path to the Utility era in Quantum Computing

The goal of this course is to learn how to implement utility-scale applications on a quantum computer. To achieve the goal, the course covers from the basics of quantum information to recent advances of quantum algorithms for noisy quantum devices as well as circuit optimization and error mitigation techniques. The course also introduces how to implement quantum algorithms using open-source framework of quantum computing and real quantum device with more than 127 qubits. The course is intended to help students understand the potential and limitations of currently available quantum devices.

Schedule: Every Friday from 16:50 to 18:20 (except May 15 (Wed), May 30(Thu))

Notes: All lectures will be held in person. Recording also will be available for reviewing.

Course Schedule 2024

Date	Lecture Title	Lecturer	Date	Lecture Title	Lecturer
4/5	Invitation to the Utility Era	Tamiya Onodera	6/7	Classical Simulation (Clifford Circuit, Tensor Network)	Yoshiaki Kawase
4/19	Quantum Gates, Circuits, and Measurements	Kifumi Numata	6/14	Quantum Hardware	Masao Tokunari / Tamiya Onodera
4/26	Quantum Teleportation / Superdense Coding	Kifumi Numata	6/21	Quantum Circuit Optimization (Transpilation)	Toshinari Itoko
5/10	Quantum Algorithms: Grover Search	Atsushi Matsuo	6/28	Quantum Noise and Quantum Error Mitigation	Toshinari Itoko
5/15 (Wed)	Quantum Algorithms: Phase Estimation	Kento Ueda	7/5	Utility Scale Experiment I	Tamiya Onodera
5/24	Quantum Algorithms: Variational Quantum Algorithms (VQA)	Takashi Imamichi	7/12	Utility Scale Experiment II	Yukio Kawashima
5/30 (Thu)	Quantum Simulation (Ising model, Heisenberg, XY model), Time Evolution (Suzuki Trotter, QDrift)	Yukio Kawashima	7/19	Utility Scale Experiment III	Kifumi Numata / Tamiya Onodera/ Toshinari Itoko

Lecture 2: Quantum Bits, Gates, and Circuits

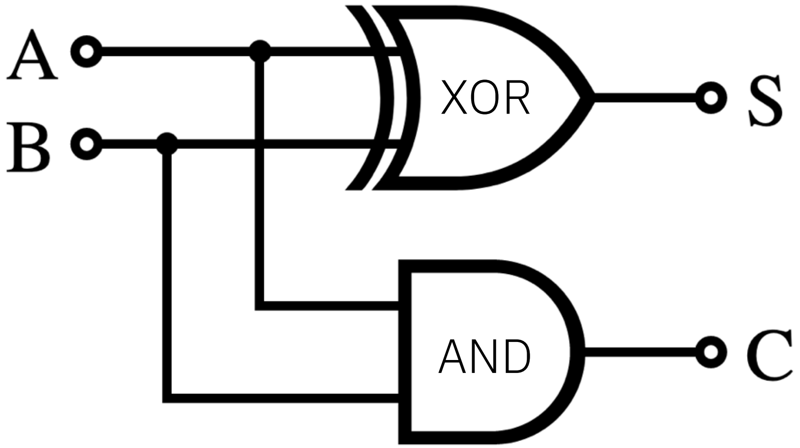
- Understanding Quantum Computation with Circuit Models using quantum bits and gates.

- Hands on using Qiskit
  - If you didn't install Qiskit in your laptop, please install it.

  https://docs.quantum.ibm.com/guides/install-qiskit

Circuits for addition in classical computing

A classical logic circuit is a set of gate operations on bits and is the unit of computation.



Half adders circuit

Truth table

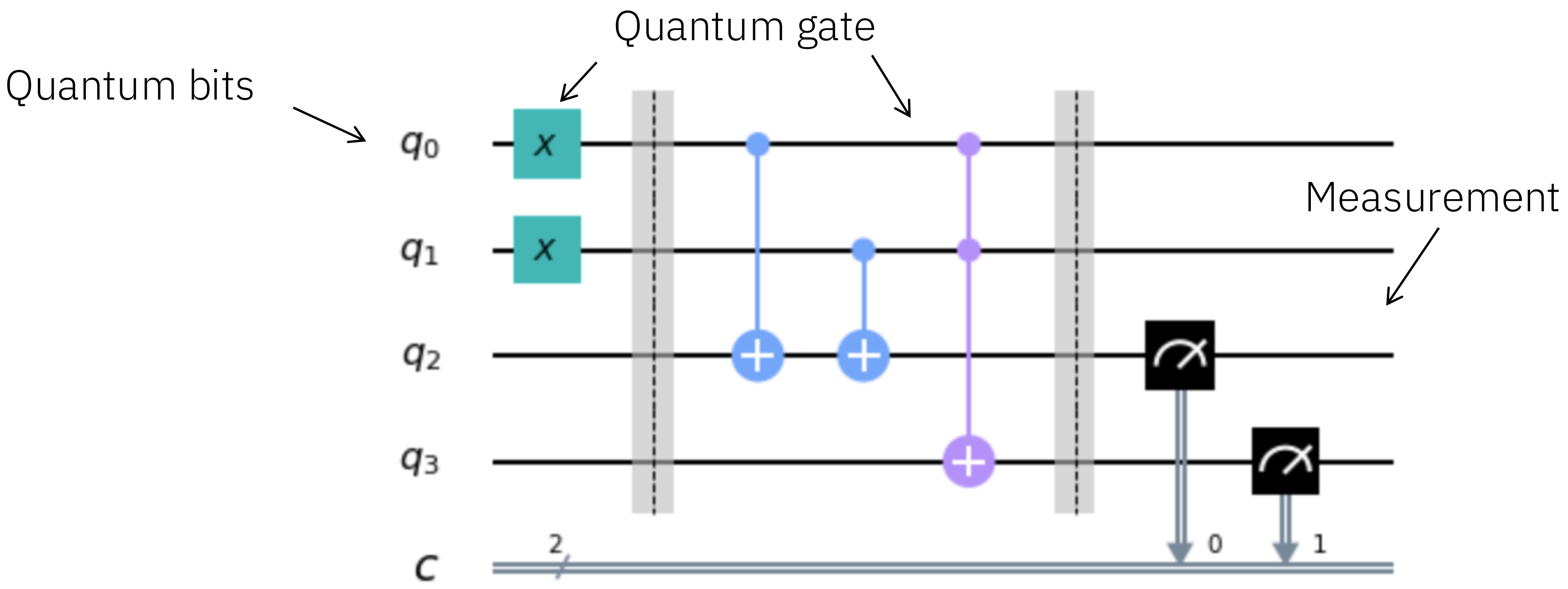
A (input)	B (input)	S (sum)	C (carry out)
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Inputs are on the left, outputs are on the right, and operations are represented by symbols between them.

Models of Quantum Computing

For quantum computers, we use the same basic idea but have different conventions for how to represent inputs, outputs, and the symbols used for operations.

- A sequence of basic quantum gates are applied on quantum bits.



Half adders circuit in quantum computing

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2. Quantum gates

Quantum gates are operations that modify the states of qubits. You will learn typical single-qubit gates, a mathematical representation of qubit state, and Bloch sphere.

Typical single-qubit gates

NOT gate

$$|0\rangle \rightarrow \boxed{X} \rightarrow |1\rangle$$

Hadamard gate

$$|0\rangle \rightarrow \boxed{H} \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

Superposition

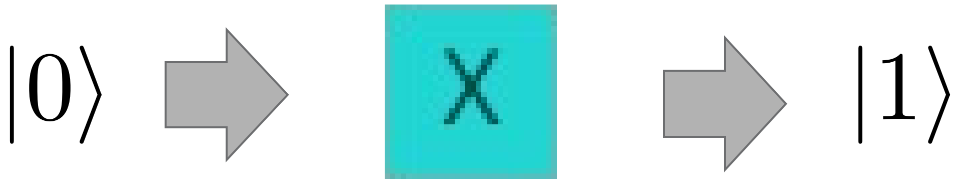
Single-qubit quantum state

$|0\rangle$ and $|1\rangle$ are vectors in the two-dimensional complex vector space $\mathbb{C}^2$:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For example, X gate is

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$X\,|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$

Single-qubit quantum state and unitary evolution

The arbitrary quantum state can be represented as a linear combination of $|0\rangle$ and $|1\rangle$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$.

The quantum state is evolved by Unitary operator U.

$$|\psi'\rangle = U|\psi\rangle$$

$$U^\dagger U = UU^\dagger = I, \qquad U^\dagger = U^{-1}$$

The quantum operation is reversible.

$$U^{-1}U|\psi\rangle = |\psi\rangle$$

Bloch Sphere

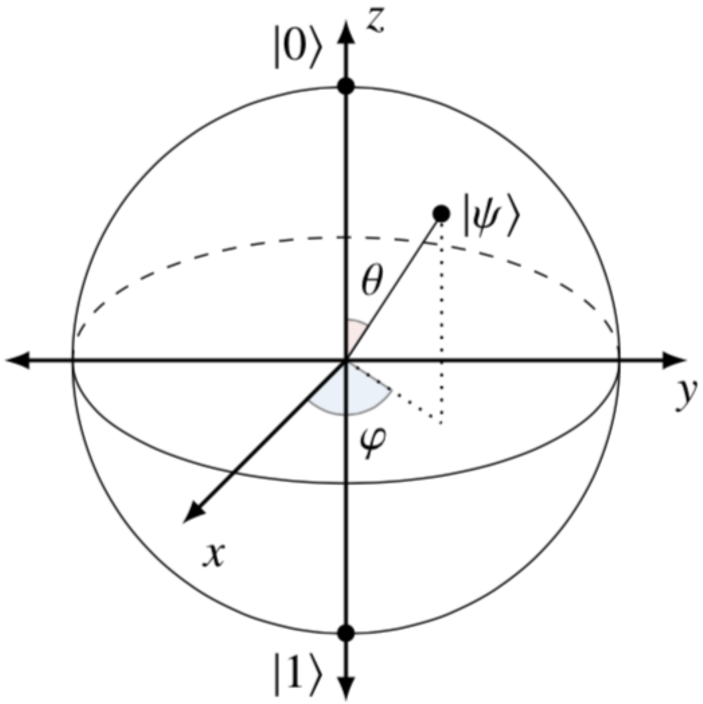
- A quantum state of single-qubit is

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad s.t. |\alpha|^2 + |\beta|^2 = 1$$

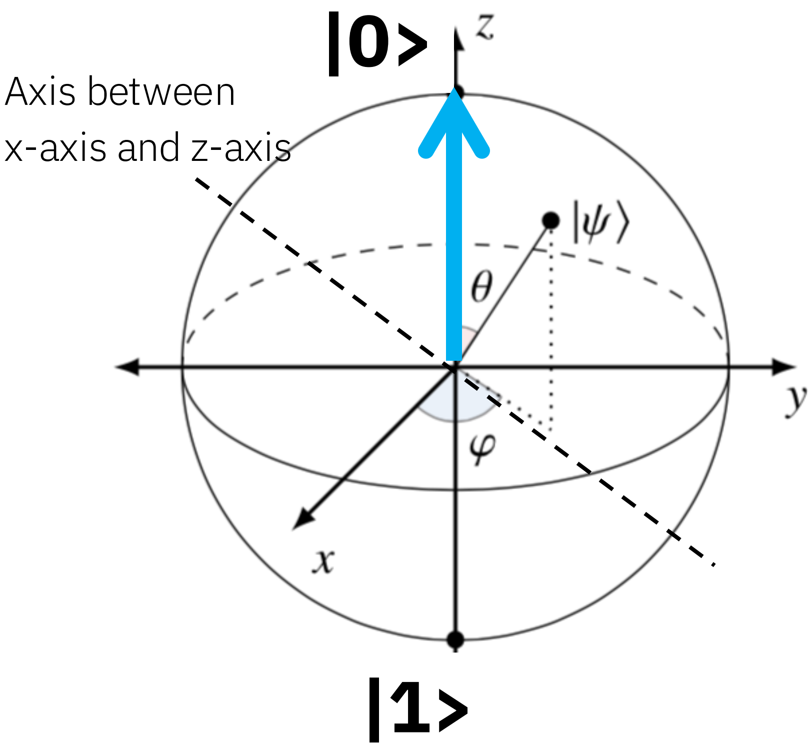
- This allows us to write the quantum state as

$$|\psi\rangle = cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi} sin\left(\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\varphi} sin\frac{\theta}{2} \end{pmatrix}$$

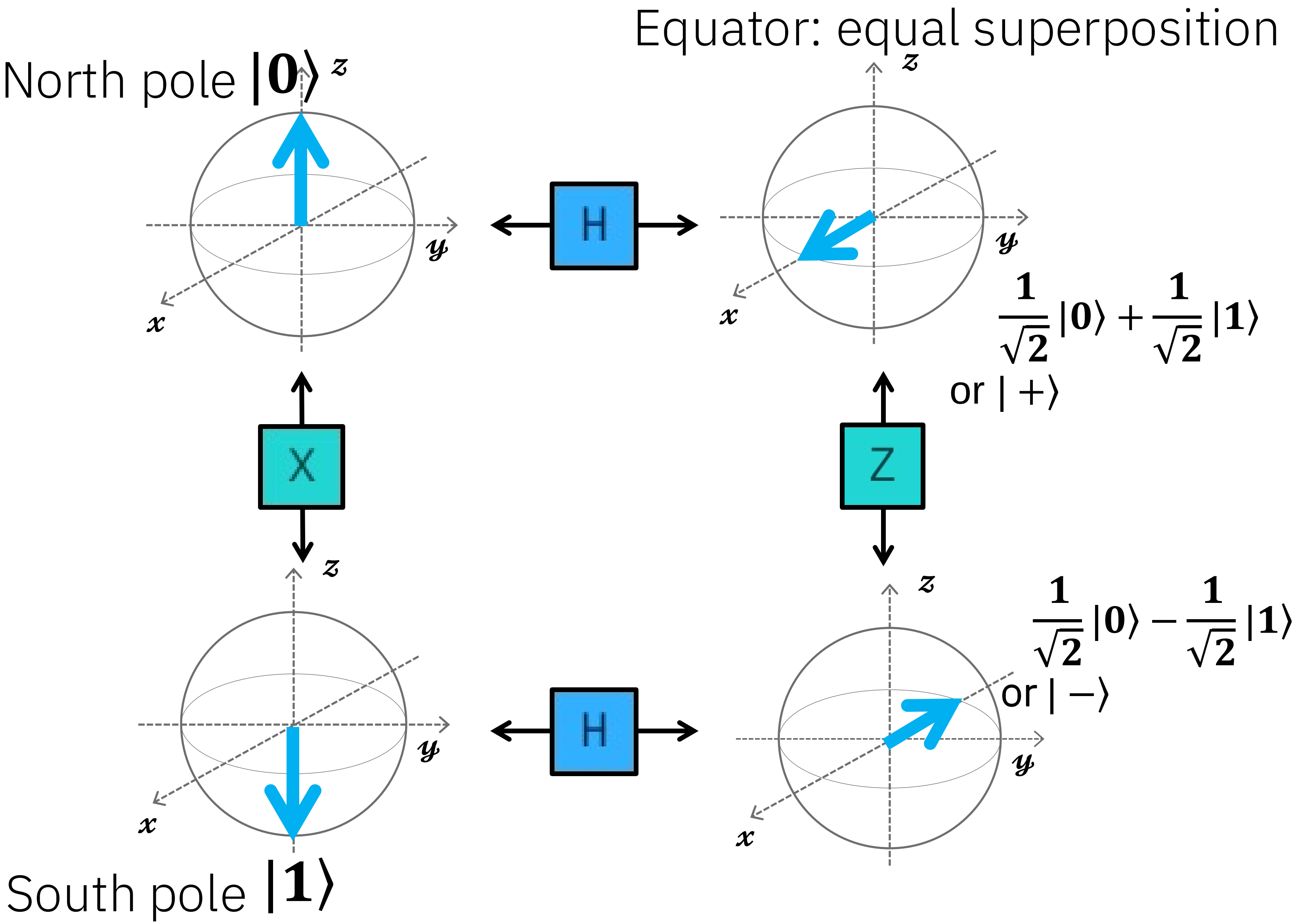
- The single qubit quantum state can be mapped to the Bloch sphere.



Bloch sphere



A pure quantum state is a vector pointing from the center to a point on the sphere of radius 1.



Typical single-qubit gates

$$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}; \quad T = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{bmatrix}.$$

$$R_x(\theta) \equiv e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_y(\theta) \equiv e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$R_z(\theta) \equiv e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

Reference: https://javafxpert.github.io/grok-bloch/

3. Superposition and measurement

Superposition is creating a quantum state that is a combination of |0> and |1>. Here, you will also learn measurement, measurement operators, and global phase.

Superposition

Superposition is creating a quantum state that is a combination of $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \qquad s.t. |\alpha|^2 + |\beta|^2 = 1$$

Note that if α and β are non-zero, then the qubit's state contains both $|0\rangle$ and $|1\rangle$.

This is what people mean when they say that a qubit can be "0 and 1 at the same time."

Measurement

Measurement is forcing the qubit's state

$$\alpha|0\rangle + \beta|1\rangle \qquad s.t. |\alpha|^2 + |\beta|^2 = 1$$

to $|0\rangle$ or $|1\rangle$ by observing it, where

$|\alpha|^2$ is the probability we will get $|0\rangle$ when we measure.

$|\beta|^2$ is the probability we will get $|1\rangle$ when we measure. (Born rule)

So, α and β are called probability amplitudes.

For example,

$\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$ has an equal probability of becoming $|0\rangle$ or $|1\rangle$, and

$\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}i|1\rangle$ has a 75% chance of becoming $|0\rangle$.

Measurement operators

In case of standard basis measurements, the measurement operators are

$$M_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; M_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Standard basis is $|0\rangle$ and $|1\rangle$.

If the state of the quantum system is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then the probabilities of observing the outcome are

$$p_0(outcome\ is\ 0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = (\alpha^*, \beta^*)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2$$

$$p_1(outcome\ is\ 1) = \langle\psi|M_1^\dagger M_1|\psi\rangle = (\alpha^*, \beta^*)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\beta|^2$$

and the quantum states after the measurement are

$$\frac{M_0|\psi\rangle}{\sqrt{\langle\psi|M_0^\dagger M_0|\psi\rangle}} = \frac{\alpha}{|\alpha|}|0\rangle \cong |0\rangle, \; \frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}} = \frac{\beta}{|\beta|}|1\rangle \cong |1\rangle$$

Global phase

Suppose that $|\psi\rangle$ and $|\phi\rangle$ are unit vectors representing quantum states, and assume that there exists a complex number α on the unit circle (meaning that $|\alpha| = 1$, or alternatively $\alpha = e^{i\theta}$ for some real number θ) such that

$$|\phi\rangle = \alpha|\psi\rangle.$$

Then, the vectors $|\psi\rangle$ and $|\phi\rangle$ are said to differ by a **global phase.**
We also refer to α as a **global phase**.

The two states are considered to be equivalent, because when we measure them, we got the same result:

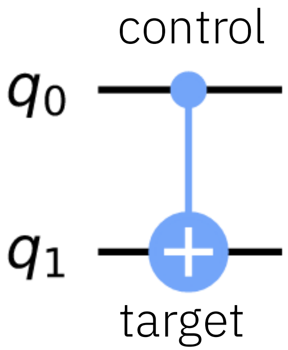
$$\langle\phi|M_j^\dagger M_j|\phi\rangle = \alpha^*\alpha\langle\psi|M_j^\dagger M_j|\psi\rangle = \langle\psi|M_j^\dagger M_j|\psi\rangle$$

For example,

- Different state: $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ and $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$
- Same state: $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ and $-|-\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

Typical two-qubit gates

CNOT gate is a conditional gate that performs an X-gate on the target qubit, if the state of the control qubit is |1>.



Truth table

Input (t,c)	Output (t,c)
00	00
01	11
10	10
11	01

Note: Qiskit uses **Little Endian**, $|q_1 q_0\rangle$

Acting on the 4D-statevector, it has one of the two matrices, depending on which qubit is the control and which is the target.

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Different books, simulators and papers order their qubits differently. In Qiskit, the left matrix corresponds to the CNOT in the circuit above.

Superposition of multiple systems

- A one-qubit system can be in the superposition of two states:

$$|0\rangle, |1\rangle$$

- A two-qubit system can be in the superposition of 2^2 states:

$$|0\rangle \otimes |0\rangle, |1\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |1\rangle$$

- An n-qubit system can be in the superposition of 2^n states:

$$|0\rangle_{n-1} \otimes \cdots \otimes |0\rangle_0, |0\rangle_{n-1} \otimes \cdots \otimes |0\rangle_1 \otimes |1\rangle_0, \cdots, |1\rangle_{n-1} \otimes \cdots \otimes |1\rangle_0$$

*Important Notations in Quantum Computing

- Tensor products

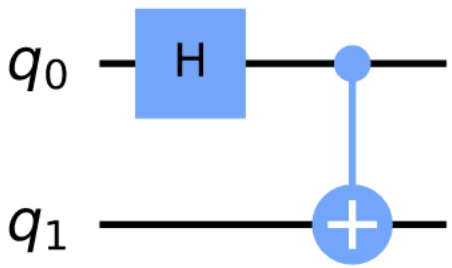
$$|0\rangle \otimes |0\rangle \equiv |0\rangle\,|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle$$

More generally,

$$\begin{pmatrix}\alpha_1\\ \dots\\ \alpha_m\end{pmatrix} \otimes \begin{pmatrix}\beta_1\\ \dots\\ \beta_n\end{pmatrix} = \begin{pmatrix}\alpha_1\beta_1\\ \dots\\ \alpha_1\beta_n\\ \dots\\ \alpha_m\beta_n\end{pmatrix}$$

Entangled state

An entangled state is a state $|\psi\rangle_{AB}$ consisting of quantum states $|\psi\rangle_A$ and $|\psi\rangle_{\mathrm{B}}$ that cannot be represented by a tensor product of individual quantum states.



$$|0\rangle\otimes|0\rangle \rightarrow H\otimes I \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|0\rangle$$
$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$
$$\rightarrow CNOT \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

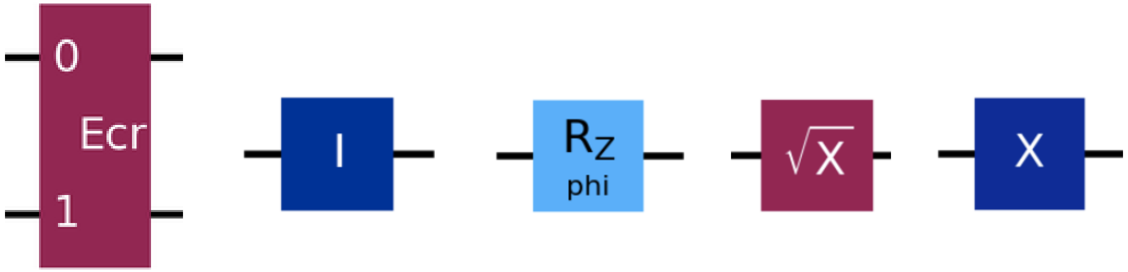
- $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a unit vector.
- However, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq (a_0|0\rangle + a_1|1\rangle)\otimes(b_0|0\rangle + b_1|1\rangle)$

There is no coefficient which satisfies this equation.

Basis gate set

Only a limited set of gates can be executed directly on the hardware.

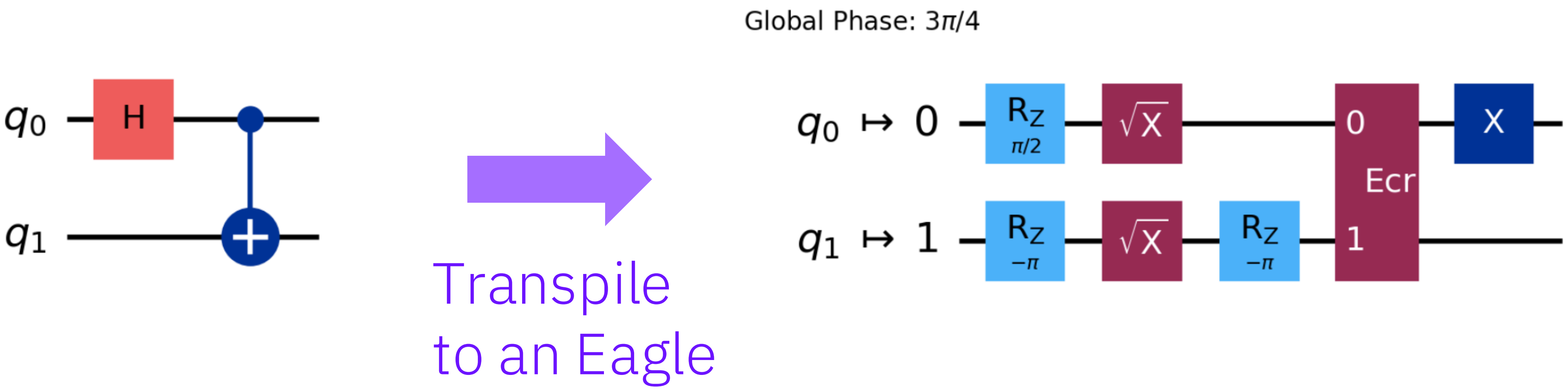
The basis gate set of an IBM Quantum Eagle processor is {ECR, ID, RZ, SX, X}.



- ECR (Echoed Cross Resonance) $= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & 1 \\ 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}$
- SX (sqrt X) $= \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

Basis gate set

Only a limited set of gates can be executed directly on the hardware.
Other gates can be transpiled into these basis gates.



8. Exercise 2 solution and closing

You can see the solution of Exercise 2.

Hands on

Lecture 2: Quantum Bits, Gates, and Circuits

- Understanding Quantum Computation with Circuit Models using quantum bits and gates.

- Hands on using Qiskit
  1. Single-qubit quantum gates
     - State vector simulator, Bloch sphere
  2. Multi-qubit quantum gates
     - Aer simulator, Real device, Qiskit Patterns
     - GHZ state of 8 qubits with the shallowest depth

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Thank you