Lesson 5. Quantum Algorithms: Phase estimation

1. Opening and phase estimation problem

Phase estimation problem is to efficiently determine the phase corresponding to the eigenvalue of the quantum state. In this class, we will first look at a simple example of phase estimation problem, then learn quantum Fourier transform, and then return to the phase estimation procedure.

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5. Quantum Algorithms: Phase estimation

2024/05/15

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Course Schedule 2024

Date	Lecture Title	Lecturer	Date	Lecture Title	Lecturer
4/5	Invitation to the Utility era	Tamiya Onodera	6/7	Classical simulation (Clifford circuit, tensor network)	Yoshiaki Kawase
4/19	Quantum Gates, Circuits, and Measurements	Kifumi Numata	6/14	Quantum Hardware	Masao Tokunari
4/26	LOCC (Quantum teleportation/superdense coding/Remote CNOT)	Kifumi Numata/ Atsushi Matsuo	6/21	Quantum Circuit Optimization (transpiler)	Toshinari Itoko
5/10	Quantum Algorithms: Grover's algorithm	Atsushi Matsuo	6/28	Pauli twirling and Noise model (Pauli Transfer Matrix) Error mitigation (PEC, ZNE (PEA))	Toshinari Itoko
5/15 (Wed)	Quantum Algorithms: Phase estimation	Kento Ueda	7/5	Quantum Utility I (127Qubit GHZ)	Kifumi Numata
5/24	Quantum Algorithms: Variational Quantum Algorithms (VQA)	Takashi Imamichi	7/12	Quantum Utility II (Utility paper implementation)	Tamiya Onodera
5/30 (Thu)	Quantum simulation (Ising model, Heisenberg, XY model), Time evolution (Suzuki Trotter, QDrift)	Yukio Kawashima	7/19	Quantum Utility III (Krylov subspace expansion)	Yukio Kawashima

Overview

- Phase estimation problem
  - Warm-up: using the phase kickback
  - Iterating the unitary operation
  - Two control qubits
  - Two-qubit phase estimation
- Quantum Fourier transform
- Phase estimation procedure

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Phase estimation problem

Phase estimation problem

Input: A unitary quantum circuit for an n-qubit operation U and an n qubit quantum state $|\psi\rangle$

Promise: $|\psi\rangle$ is an eigenvector of U

Output: An approximation to the number $\theta \in [0, 1)$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

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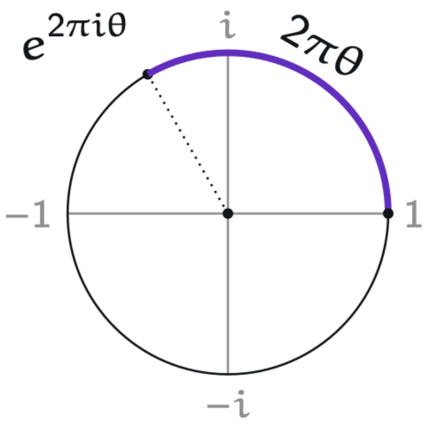

We can approximate θ by a fraction

$$\theta \approx \frac{y}{2^m}$$

for $y \in \{0, 1, \ldots, 2^m - 1\}$.

This approximation is taken "modulo 1."

Phase estimation problem

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The Phase estimation problem is applied to the energy calculations for quantum many-body systems and Shor's algorithm.

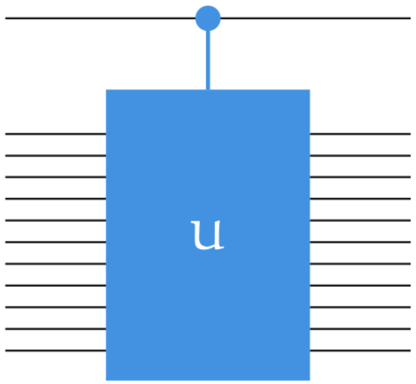
e.g. Shor’s algorithm : Acceleration from quasi-exponential to polynomial time.

2. Phase kickback and iterating the unitary operation

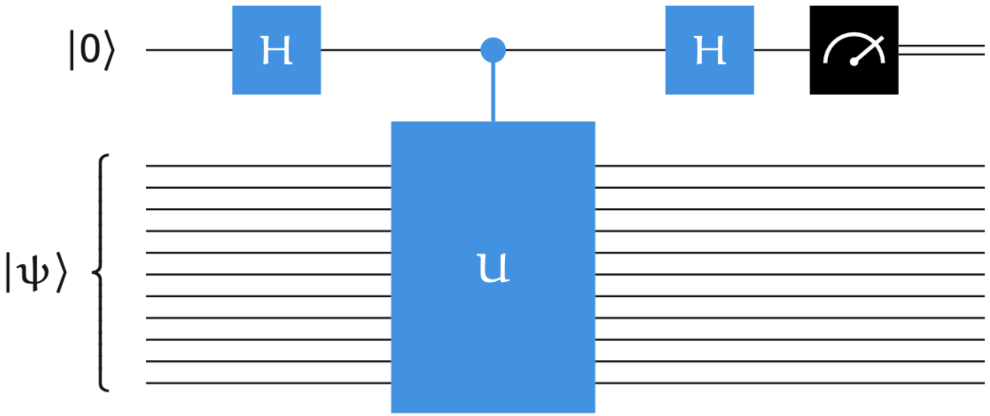
To understand phase estimation, you must first learn the phase kickback circuit using a controlled-U gate. This circuit provides low-precision solutions to the phase-estimation problem. Operation of the controlled-U gate twice gives us more information about the phase.

Warm-up: using the phase kickback

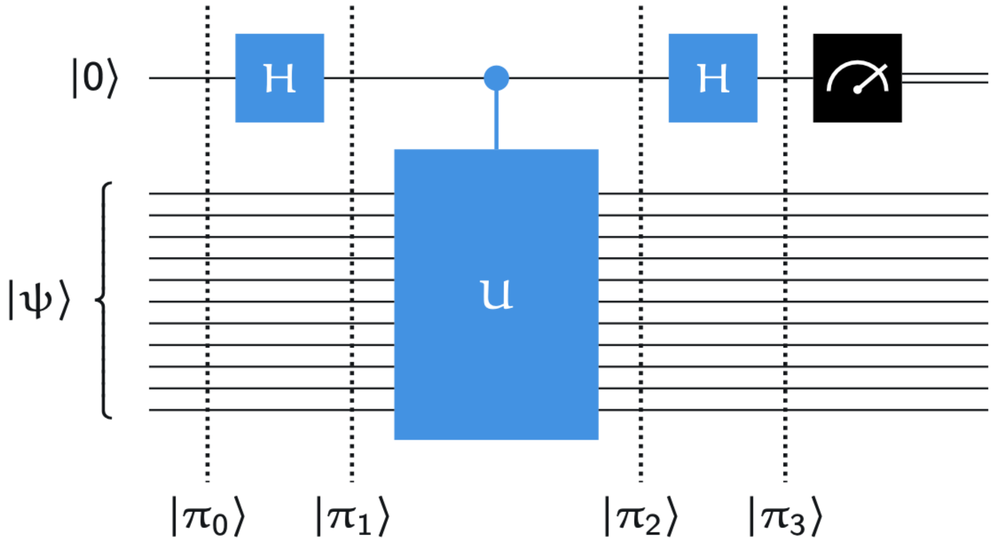
Given a circuit for $\mathcal{U}$, we can create a circuit for a controlled-$\mathcal{U}$ operation:



Let's consider this circuit:



Warm-up: using the phase kickback

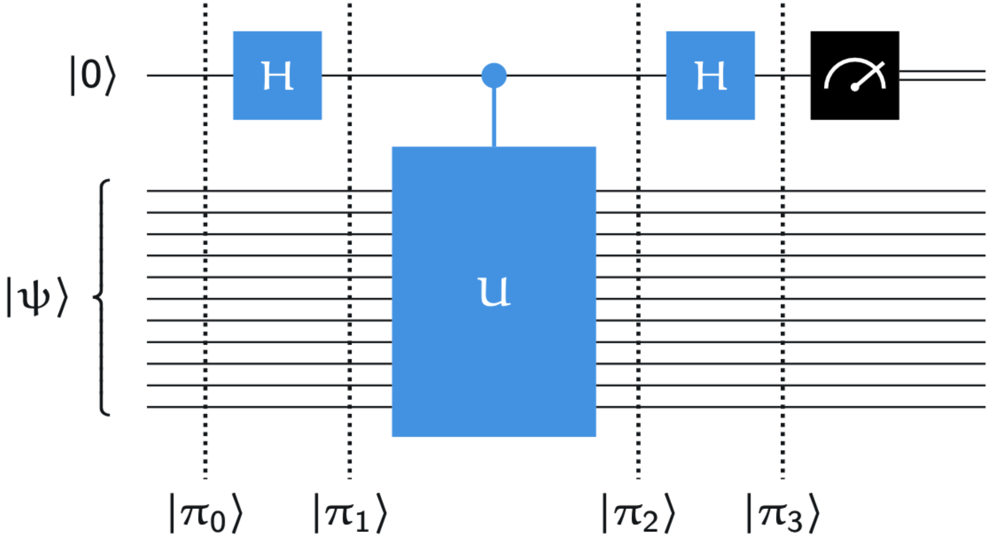


$$|\pi_0\rangle = |\psi\rangle|0\rangle$$

$$|\pi_1\rangle = \frac{1}{\sqrt{2}}|\psi\rangle|0\rangle + \frac{1}{\sqrt{2}}|\psi\rangle|1\rangle$$

$$|\pi_2\rangle = \frac{1}{\sqrt{2}}|\psi\rangle|0\rangle + \frac{1}{\sqrt{2}}(U|\psi\rangle)|1\rangle = |\psi\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{2\pi i\theta}}{\sqrt{2}}|1\rangle\right)$$

Warm-up: using the phase kickback



$$|\pi_2\rangle = |\psi\rangle \otimes \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{e^{2\pi i\theta}}{\sqrt{2}} |1\rangle \right)$$

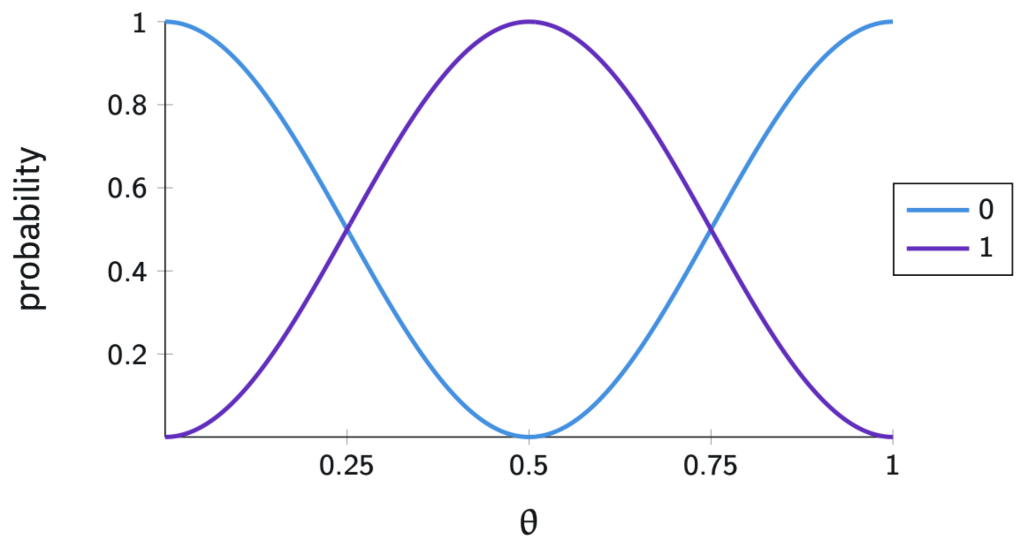
$$|\pi_3\rangle = |\psi\rangle \otimes \left(\frac{1 + e^{2\pi i\theta}}{2} |0\rangle + \frac{1 - e^{2\pi i\theta}}{2} |1\rangle \right)$$

Warm-up: using the phase kickback

$$|\psi\rangle \otimes \left(\frac{1 + e^{2\pi i \theta}}{2} |0\rangle + \frac{1 - e^{2\pi i \theta}}{2} |1\rangle \right)$$

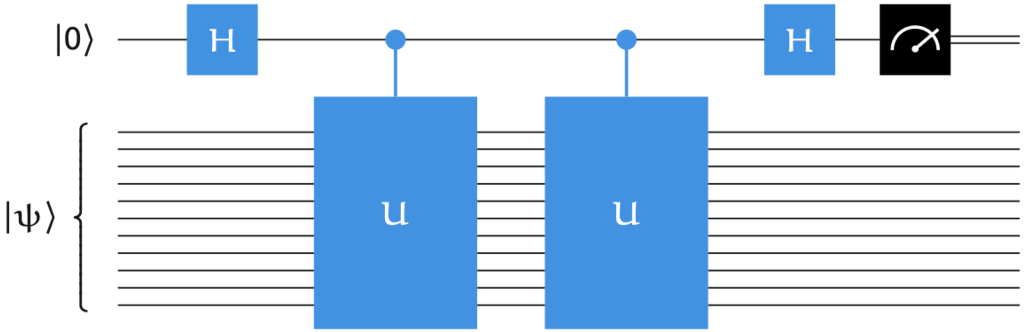
Measuring the top qubit yields the outcomes 0 and 1 with these probabilities:

$$p_0 = \left| \frac{1 + e^{2\pi i \theta}}{2} \right|^2 = \cos^2(\pi\theta) \qquad p_1 = \left| \frac{1 - e^{2\pi i \theta}}{2} \right|^2 = \sin^2(\pi\theta)$$

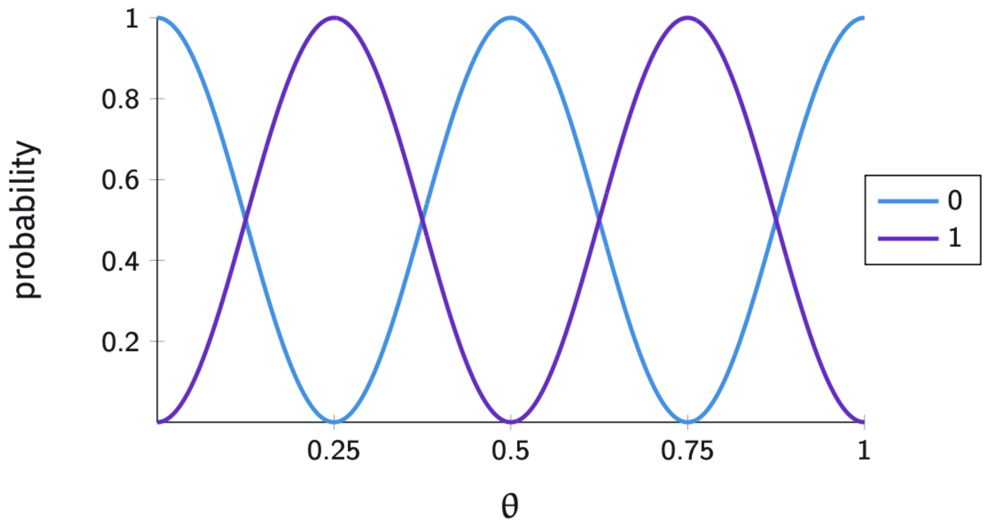


Iterating the unitary operation

How can we learn more about θ? One possibility is to apply the controlled-$\mathcal{U}$ operation twice (or multiple times):



Performing the controlled-$\mathcal{U}$ operation twice has the effect of squaring the eigenvalue:



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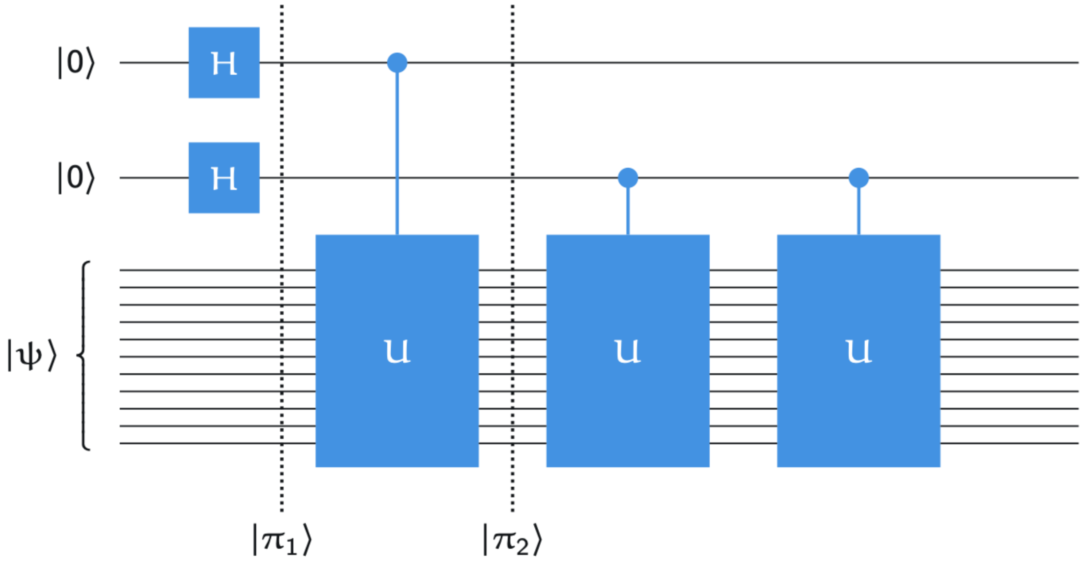
3. Two-qubit phase estimation

By extending the circuit to two control qubits circuit and adding the inverse of quantum Fourier transform circuit, you can create a two-qubit phase estimation circuit. You will learn the concept of phase estimation from this procedure.

Two control qubits

Let's use two control qubits to perform the controlled-U operations — and then we'll see how best to proceed.

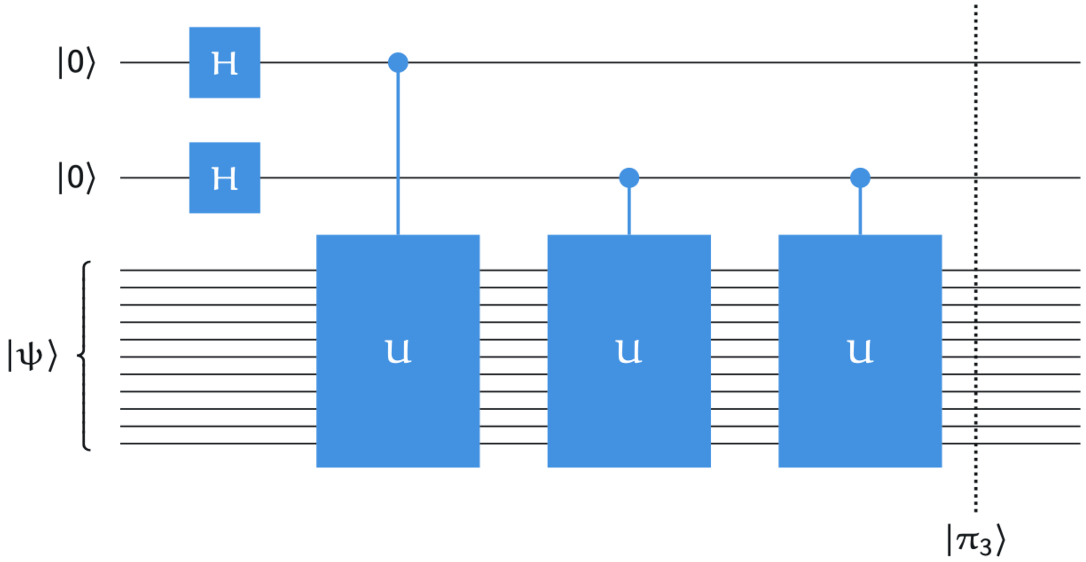


$$|\pi_1\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} |a_1 a_0\rangle$$

$$|\pi_2\rangle = |\psi\rangle \otimes \frac{1}{2} \sum_{a_0=0}^{1} \sum_{a_1=0}^{1} e^{2\pi i a_0 \theta} |a_1 a_0\rangle$$

Two control qubits

Let's use two control qubits to perform the controlled-U operations — and then we'll see how best to proceed.



$$|\pi_3\rangle = |\psi\rangle \otimes \frac{1}{2}\sum_{a_0=0}^{1}\sum_{a_1=0}^{1} e^{2\pi i(2a_1+a_0)\theta}|a_1 a_0\rangle$$

$$= |\psi\rangle \otimes \frac{1}{2}\sum_{x=0}^{3} e^{2\pi i x\theta}|x\rangle$$

$$\sum_{a_0=0}^{1}\sum_{a_1=0}^{1} e^{2\pi i(2a_1+a_0)\theta}\,|a_1 a_0\rangle.$$

$$= |0\rangle + e^{2\pi i*1\theta}|1\rangle + e^{2\pi i*2\theta}|2\rangle + e^{2\pi i*3\theta}|3\rangle$$

Two control qubits

$$\frac{1}{2}\sum_{x=0}^{3} e^{2\pi i x\theta}|x\rangle$$

What can we learn about θ from this state? Suppose we're promised that $\theta = \frac{y}{4}$ for $y \in \{0, 1, 2, 3\}$. Can we figure out which one it is?

Define a two-qubit state for each possibility:

$$|\phi_y\rangle = \frac{1}{2}\sum_{x=0}^{3} e^{2\pi i \frac{xy}{4}}|x\rangle$$

$$|\phi_0\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$$

$$|\phi_1\rangle = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$$

$$|\phi_2\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle$$

$$|\phi_3\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{i}{2}|3\rangle$$

These vectors are *orthonormal* — so they can be discriminated perfectly by a projective measurement.

Two control qubits

$$|\phi_y\rangle = \frac{1}{2}\sum_{x=0}^{3} e^{2\pi i \frac{xy}{4}} |x\rangle$$

$$|\phi_0\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{2}|3\rangle$$

$$|\phi_1\rangle = \frac{1}{2}|0\rangle + \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle - \frac{i}{2}|3\rangle$$

$$|\phi_2\rangle = \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle - \frac{1}{2}|3\rangle$$

$$|\phi_3\rangle = \frac{1}{2}|0\rangle - \frac{i}{2}|1\rangle - \frac{1}{2}|2\rangle + \frac{i}{2}|3\rangle$$

$$V = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

The unitary matrix V whose *columns* are $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$, $|\phi_3\rangle$ has this action:

$$V|y\rangle = |\phi_y\rangle \quad \text{(for every } y \in \{0, 1, 2, 3\}\text{)}$$

We can identify y by performing the inverse of V then a standard basis measurement.

$$V^\dagger|\phi_y\rangle = |y\rangle \quad \text{(for every } y \in \{0, 1, 2, 3\}\text{)}$$

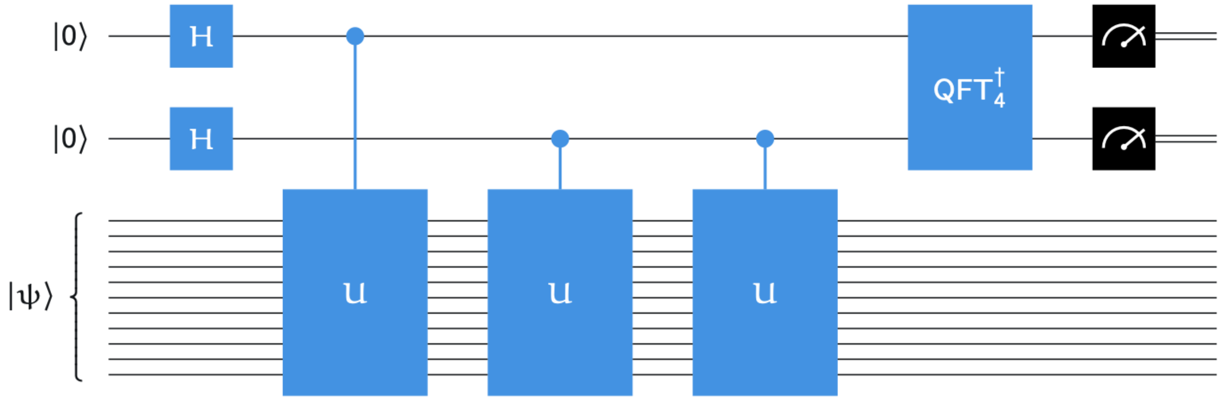
Two-qubit phase estimation

$$\mathrm{QFT}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

This matrix is associated with the *discrete Fourier transform* (for 4 dimensions).

When we think about this matrix as a unitary operation, we call it the *quantum Fourier transform.*

The complete circuit for learning $y \in \{0, 1, 2, 3\}$ when $\theta = y/4$:



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4. Quantum Fourier transform

The phase estimation problem is solved using the quantum Fourier transform. QFT transform between the computational basis and the Fourier basis. Here, you will learn the detail procedure of QFT.

Overview

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- Quantum Fourier transform

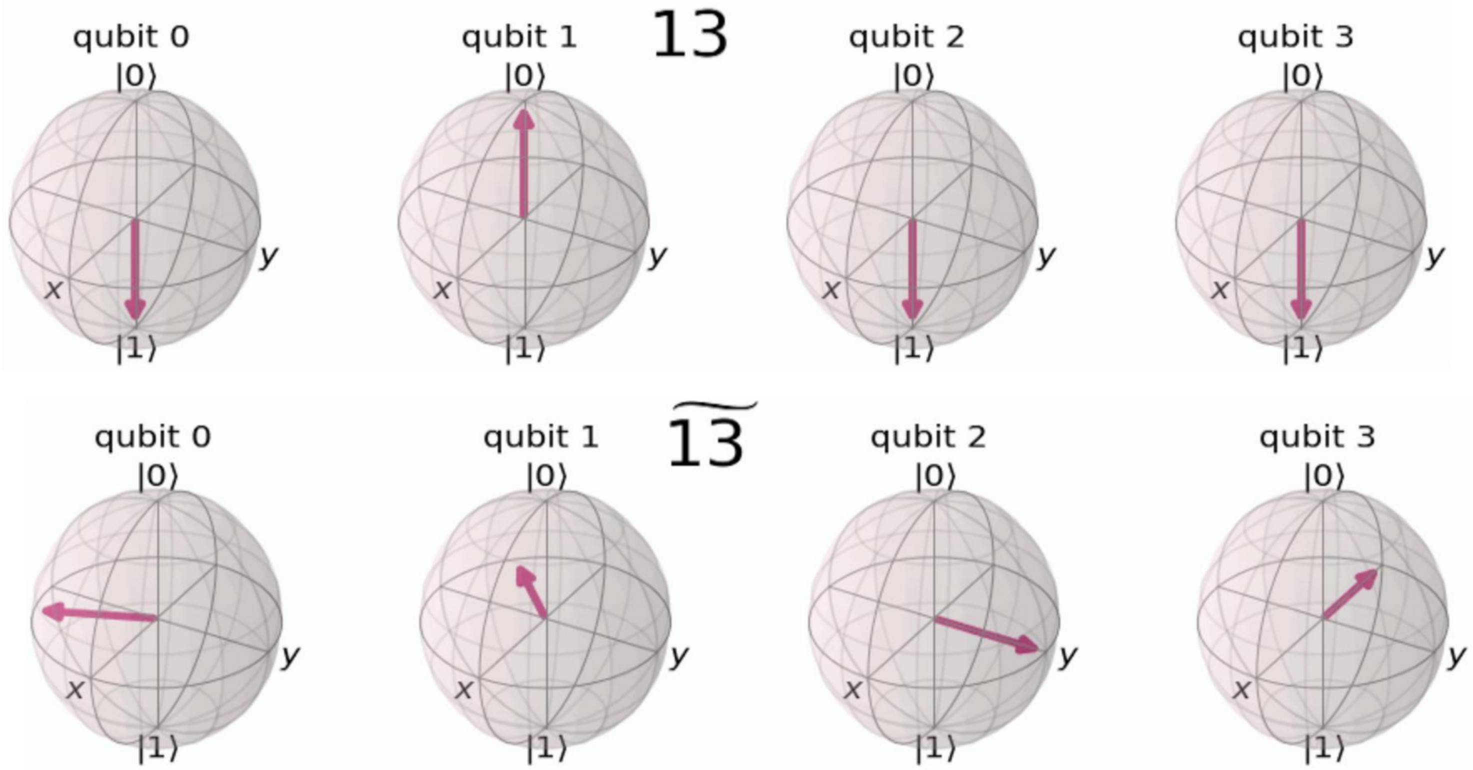
- Phase estimation procedure

Quantum Fourier transform

The quantum Fourier transform transform between the computational basis and the Fourier basis.

$$|\text{State in Computational Basis}\rangle \xrightarrow{\text{QFT}} |\text{State in Fourier Basis}\rangle$$

$$\text{QFT}|x\rangle = |\tilde{x}\rangle$$



Quantum Fourier transform

The quantum Fourier transform is defined for each positive integer N as follows.

$$\mathrm{QFT}_N = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle\langle y|$$

$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

Example

$$\mathrm{QFT}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$$

Quantum Fourier transform

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Example

$$\mathrm{QFT}_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

Quantum Fourier transform

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$$\mathrm{QFT}_N |y\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{2\pi i \frac{xy}{N}} |x\rangle$$

Example

$$\mathrm{QFT}_8 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Quantum Fourier transform

$$\mathrm{QFT_N}\,|j\rangle = \frac{1}{2^{n/2}}\sum_{k=0}^{2^n-1} exp\,(2\pi i\,jk/2^n)\,|k\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1} e^{2\pi i j\left(\sum_{l=1}^{n} k_l 2^{-l}\right)}|k_1 k_2 \ldots k_n\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1}\prod_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} \bigotimes_{m=1}^{n}|k_m\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1}\bigotimes_{m=1}^{n} e^{2\pi i j k_m 2^{-m}}|k_m\rangle$$

$$= \frac{1}{2^{n/2}}\bigotimes_{m=1}^{n}\left(|0\rangle + e^{2\pi i j 2^{-m}}|1\rangle\right)$$

$$= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\right)}{\sqrt{2}} \otimes \ldots \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_2\ldots j_n}|1\rangle\right)}{\sqrt{2}} \otimes \frac{\left(|0\rangle + e^{2\pi i 0.j_1 j_2\ldots j_n}|1\rangle\right)}{\sqrt{2}}$$

$$k = k_1 k_2 \ldots k_n = k_1 2^{n-1} + k_2 2^{n-2} + \ldots + k_n 2^0 = \sum_{l=1}^{n} k_l 2^{n-l}$$

$$k/2^n = 0.k_1 k_2 \ldots k_n = k_1 2^{-1} + k_2 2^{-2} + \ldots + k_n 2^{-n} = \sum_{l=1}^{n} k_l 2^{-l}$$

Quantum Fourier transform

$$\mathrm{QFT_N}\,|j\rangle = \frac{1}{2^{n/2}}\sum_{k=0}^{2^n-1} exp\,(2\pi i jk/2^n)\,|k\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1} e^{2\pi i j\left(\sum_{l=1}^{n} k_l 2^{-l}\right)}|k_1k_2\ldots k_n\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1}\prod_{l=1}^{n} e^{2\pi i jk_l 2^{-l}}\bigotimes_{m=1}^{n}|k_m\rangle$$

$$= \frac{1}{2^{n/2}}\sum_{k_1=0}^{1}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{1}\bigotimes_{m=1}^{n} e^{2\pi i jk_m 2^{-m}}|k_m\rangle$$

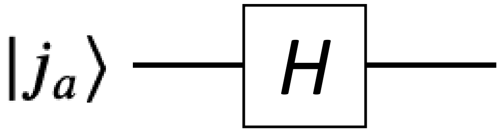
$$= \frac{1}{2^{n/2}}\bigotimes_{m=1}^{n}\left(|0\rangle + e^{2\pi i j 2^{-m}}|1\rangle\right)$$

$$= \frac{\left(|0\rangle + e^{2\pi i 0.j_n}|1\rangle\right)}{\sqrt{2}}\otimes\frac{\left(|0\rangle + e^{2\pi i 0.j_{n-1}j_n}|1\rangle\right)}{\sqrt{2}}\otimes\ldots\otimes\frac{\left(|0\rangle + e^{2\pi i 0.j_2\ldots j_n}|1\rangle\right)}{\sqrt{2}}\otimes\frac{\left(|0\rangle + e^{2\pi i 0.j_1j_2\ldots j_n}|1\rangle\right)}{\sqrt{2}}$$

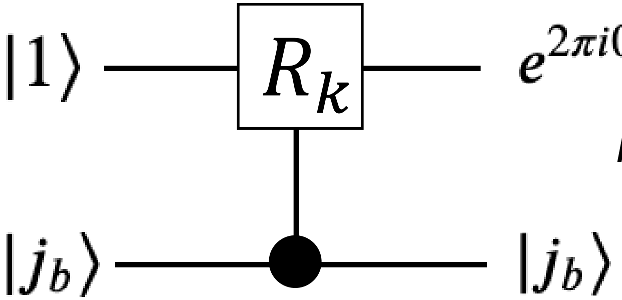
$$j2^{-m} = j_1\ldots j_{n-m}.\,j_{n-m+1}\ldots j_n$$

$$e^{2\pi i j 2^{-m}} = e^{2\pi i 0.j_{n-m+1}\ldots j_n}$$

Quantum Fourier transform



$$\frac{\left(|0\rangle + e^{2\pi i 0.j_a}|1\rangle\right)}{\sqrt{2}}$$

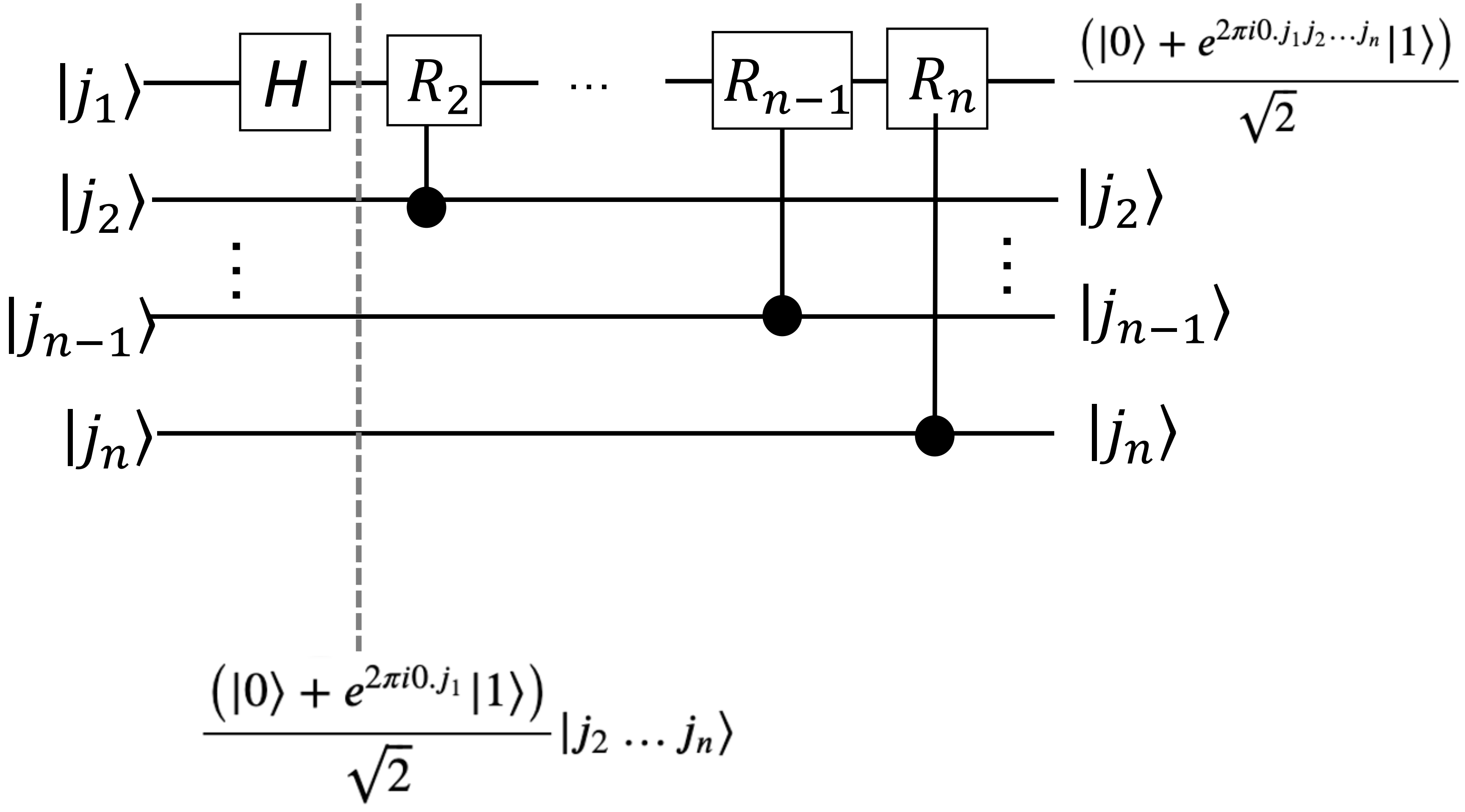


$$e^{2\pi i 0.0\ldots j_b}|1\rangle$$

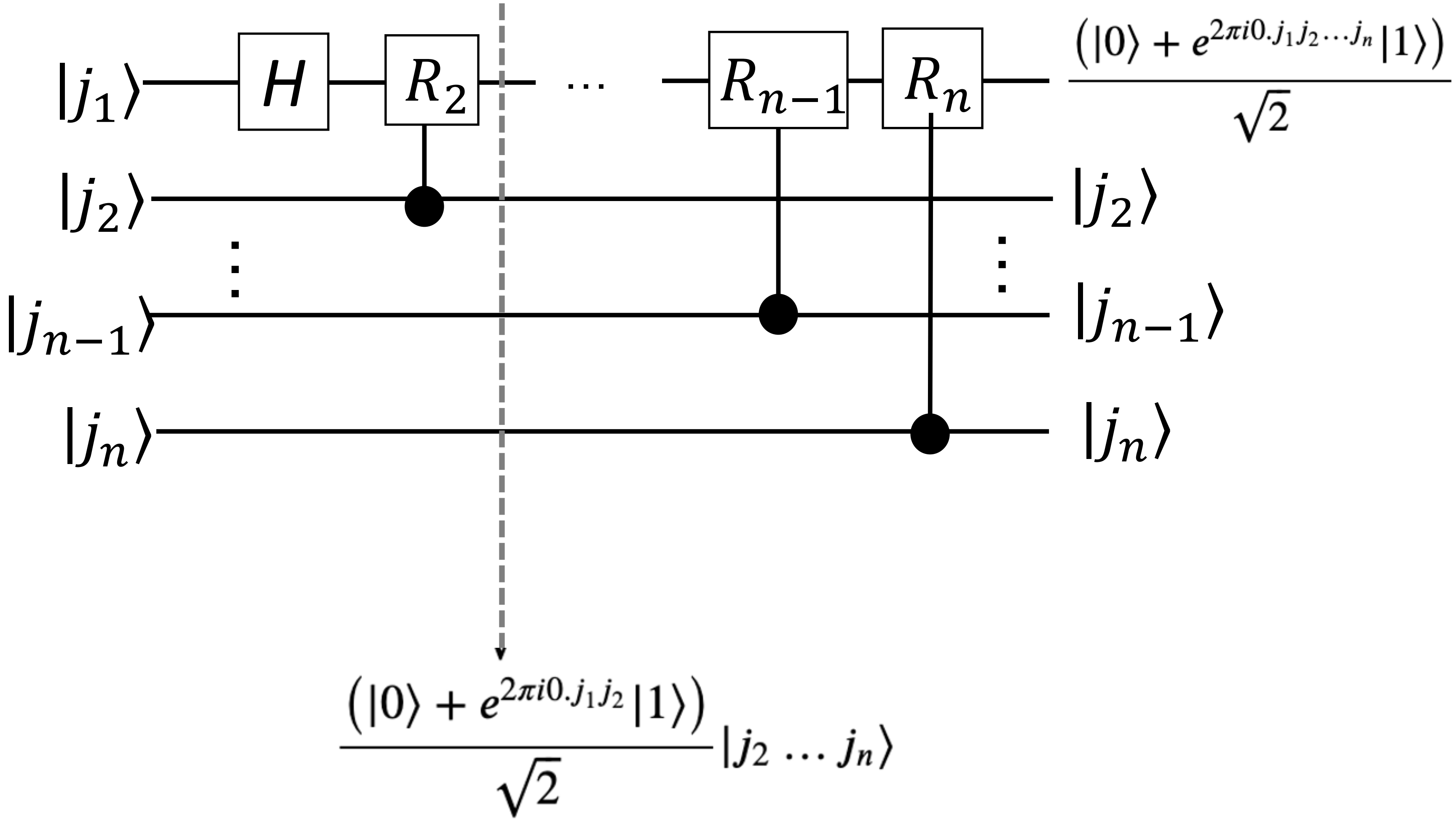
k digits in fraction part

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(\frac{2\pi i}{2^k}\right) \end{bmatrix}$$

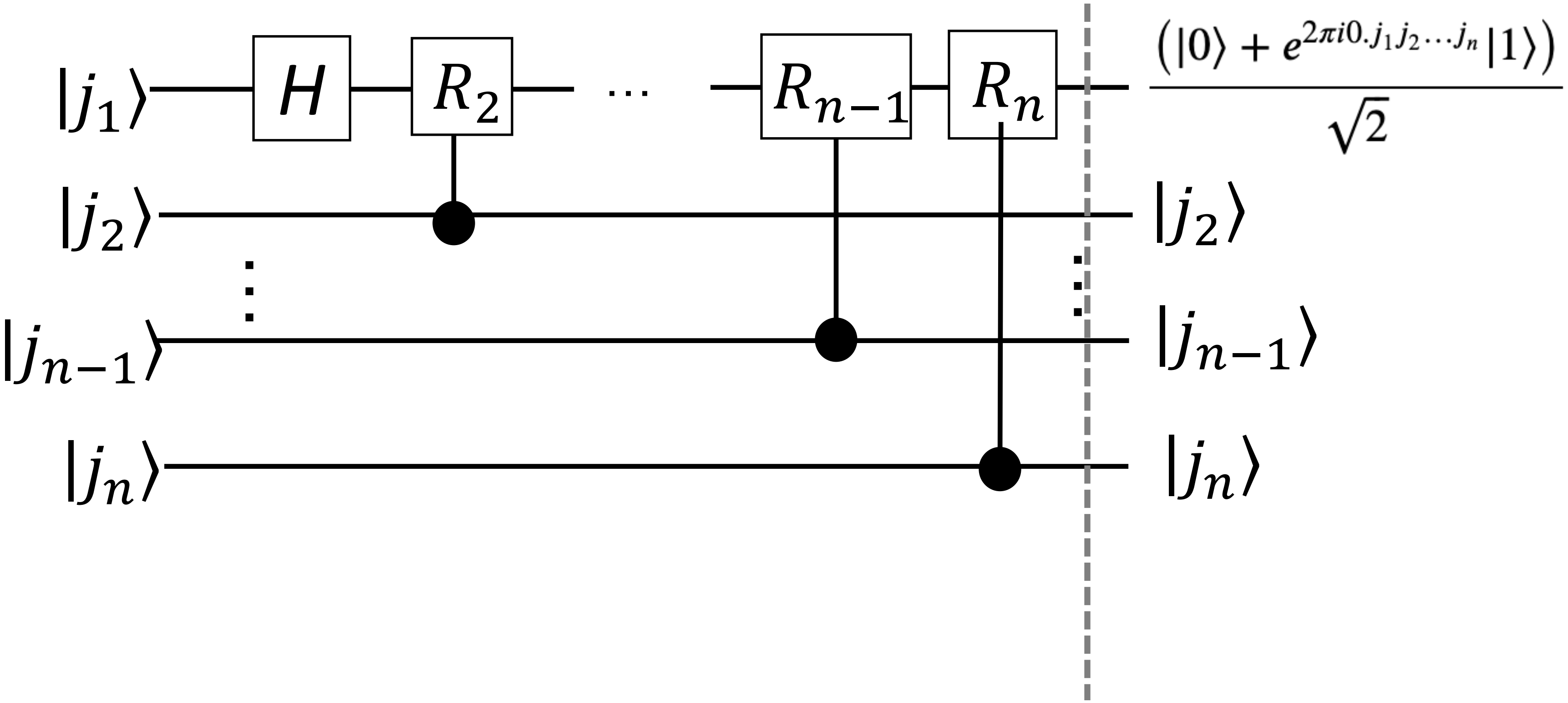
Quantum Fourier transform



Quantum Fourier transform



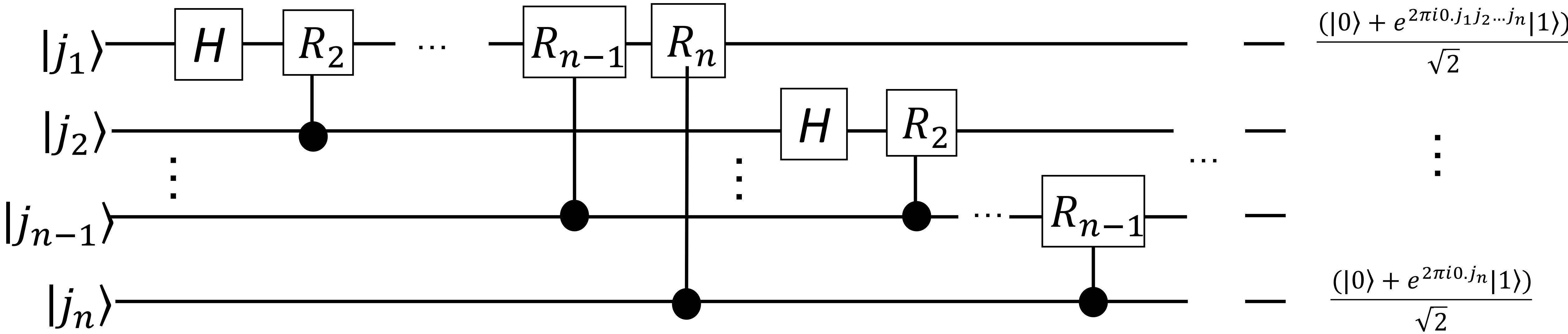
Quantum Fourier transform



$$\frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \ldots j_n}|1\rangle)}{\sqrt{2}} \; |j_2, \ldots j_n\rangle$$

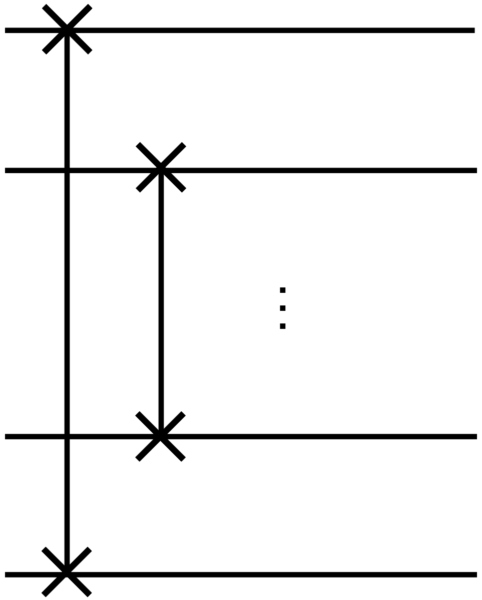
Quantum Fourier transform

The implementation is recursive in nature.



Quantum Fourier transform

At last, Swap operations are executed.



Input states (top to bottom):

$$\frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\vdots$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}}$$

Output states (top to bottom):

$$\frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}}$$

$$\vdots$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

$$\mapsto \quad \frac{(|0\rangle + e^{2\pi i 0.j_n}|1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i 0.j_{n-1} j_n}|1\rangle)}{\sqrt{2}} \otimes \cdots \otimes \frac{(|0\rangle + e^{2\pi i 0.j_2 \dots j_n}|1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle + e^{2\pi i 0.j_1 j_2 \dots j_n}|1\rangle)}{\sqrt{2}}$$

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5. Phase estimation procedure

Generalizing to the n-qubit case, you will learn the quantum phase estimation procedure in detail. You will also learn the relationship between the phase accuracy required by quantum phase estimation and the number of qubits.

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  - Warm-up: using the phase kickback
  - Iterating the unitary operation
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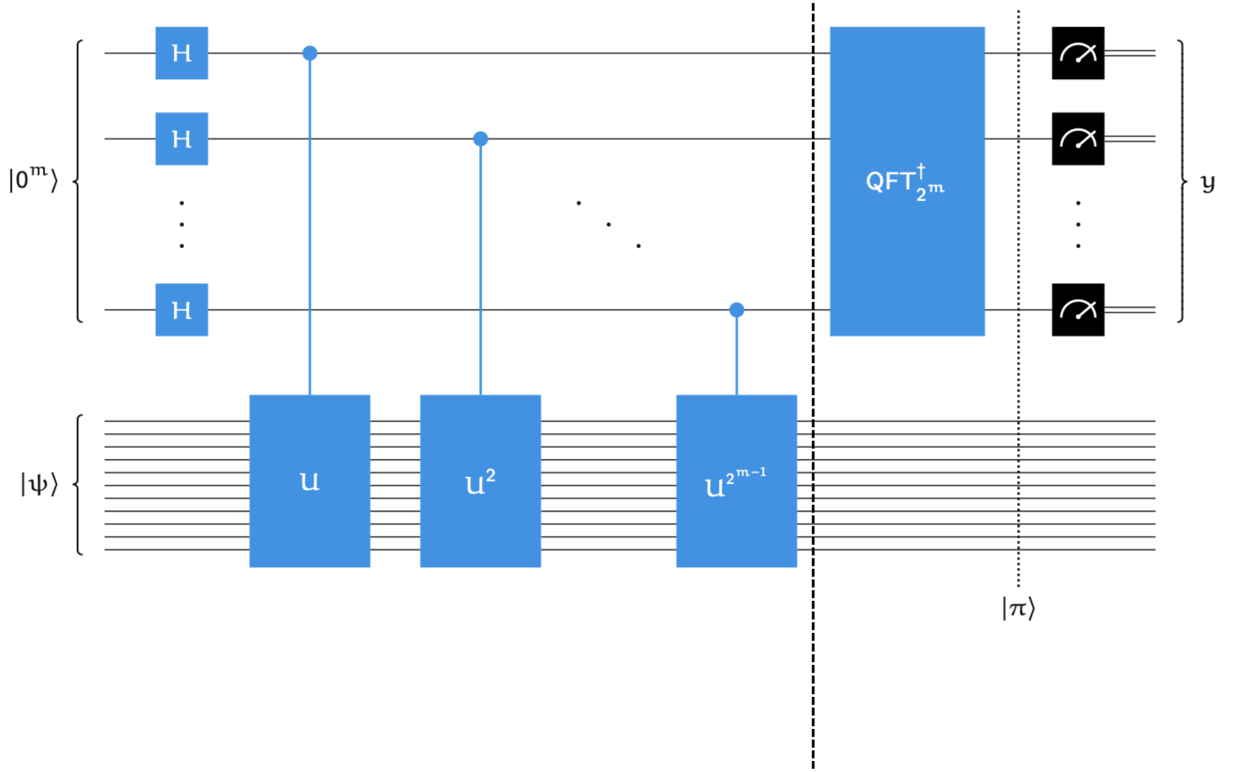
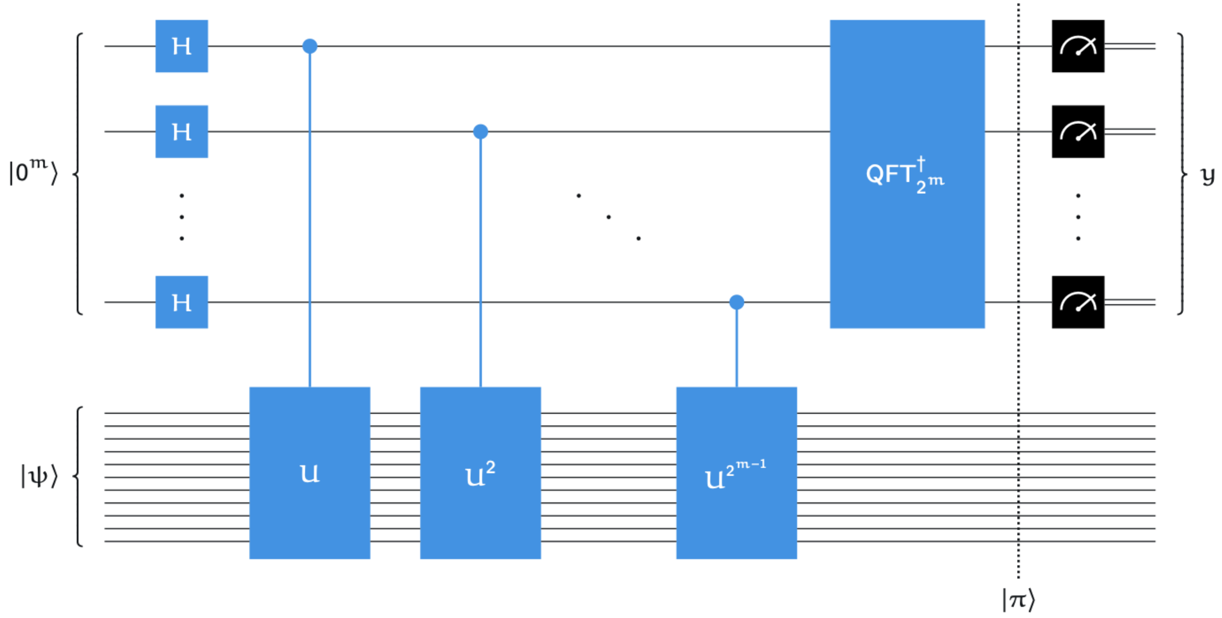
- Phase estimation procedure

Phase estimation procedure



$$|\psi\rangle \otimes \frac{1}{2^{m/2}} \sum_{x=0}^{2^m-1} e^{2\pi i x\theta} |x\rangle$$

Phase estimation procedure



$$|\pi\rangle = |\psi\rangle \otimes \frac{1}{2^m} \sum_{y=0}^{2^m-1} \sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)} |y\rangle$$

$$p_y = \left| \frac{1}{2^m} \sum_{x=0}^{2^m-1} e^{2\pi i x(\theta - y/2^m)} \right|^2$$

Phase estimation procedure

Best approximations

Suppose $y/2^m$ is the *best approximation* to θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \leq 2^{-(m+1)}$$

Then the probability to measure y will relatively high:

$$p_y \geq \frac{4}{\pi^2} \approx 0.405$$

Worse approximations

Suppose there is a *better approximation* to θ between $y/2^m$ and θ:

$$\left|\theta - \frac{y}{2^m}\right|_1 \geq 2^{-m}$$

Then the probability to measure y will be relatively low:

$$p_y \leq \frac{1}{4}$$

To obtain an approximation $y/2^m$ that is *very likely* to satisfy

$$\left|\theta - \frac{y}{2^m}\right|_1 < 2^{-m}$$

we can run the phase estimation procedure using m control qubits *several times* and take y to be the *mode* of the outcomes.

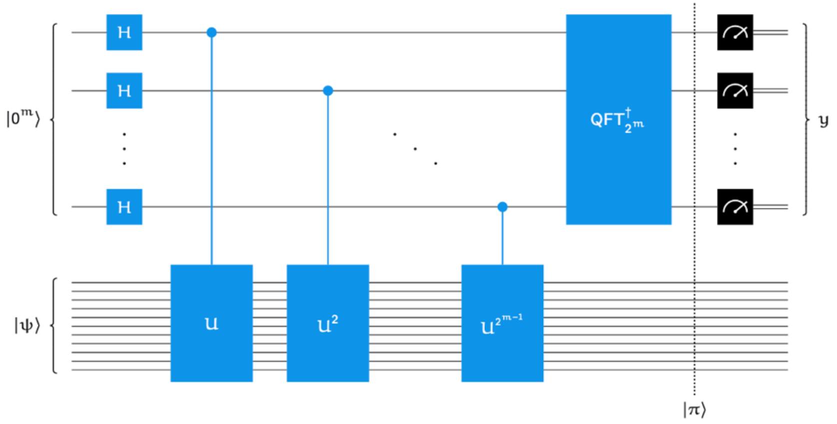
(The eigenvector $|\psi\rangle$ is unchanged by the procedure and can be reused as many times as needed.)

Summary

- The Phase estimation problem can find the approximation to the number $\theta \in [0,1)$ satisfying

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

- QFT transform between the computational basis and the Fourier basis.
- QFT can be implemented in the quantum circuit.
- The Phase estimation problem is solved using the QFT.



Reference

- John Watrous, https://learning.quantum.ibm.com/course/fundamentals-of-quantum-algorithms/phase-estimation-and-factoring .

Thank you