東京大学
THE UNIVERSITY OF TOKYO

Lesson 11. Quantum Noise and Quantum Error Mitigation

1. What is quantum noise/error

During the computation on a real quantum computer, we encounter noises. Classify the various types of noise source present in quantum computers and learn about incoherent error and coherent error.

情報科学科特別講義Ⅱ/量子計算論
Quantum noise and error mitigation

2024/06/28
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Course Schedule 2024 (subject to change)

Date	Lecture Title	Lecturer	Date	Lecture Title	Lecturer
4/5	Invitation to the Utility era	Tamiya Onodera	6/7	Classical simulation (Clifford circuit, tensor network)	Yoshiaki Kawase
4/19	Quantum Gates, Circuits, and Measurements	Kifumi Numata	6/14	Quantum Hardware	Masao Tokunari
4/26	LOCC (Quantum teleportation/superdense coding/Remote CNOT)	Kifumi Numata/ Atsushi Matsuo	6/21	Quantum circuit optimization (transpilation)	Toshinari Itoko
5/10	Quantum Algorithms: Grover's algorithm	Atsushi Matsuo	6/28	Quantum noise and quantum error mitigation	Toshinari Itoko
5/15 (Wed)	Quantum Algorithms: Phase estimation	Kento Ueda	7/5	Utility Scale Experiment I	Tamiya Onodera
5/24	Quantum Algorithms: Variational Quantum Algorithms (VQA)	Takashi Imamichi	7/12	Utility Scale Experiment II	Yukio Kawashima
5/30 (Thu)	Quantum simulation (Ising model, Heisenberg, XY model), Time evolution (Suzuki Trotter, QDrift)	Yukio Kawashima	7/19	Utility Scale Experiment III	Kifumi Numata

What you learn today

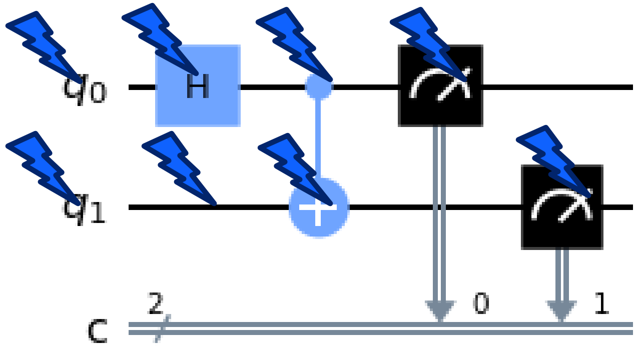
- Talk 1 (Basic, 35min)
  - What is quantum noise/error
  - Error suppression and mitigation techniques
    - TREX (Twirled Readout Error eXtinction)
    - ZNE (Zero Noise Extrapolation)
    - PEA (Probabilistic Error Amplification)

<Break>

- Hands-on (20 min)
- Talk 2 (Advanced, 30min)
  - Formalism of quantum errors
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
    - PTM (Pauli Transfer Matrix) representation

Fight noise after avoiding it as possible

- Noises everywhere:
  - Initialization
  - Gates (even in idling time)
  - Measurements



- **Noises** cause (computational) **errors**
- Errors prevent the realization of useful quantum computers

Quantum circuit optimization (last week) ➔ Reduce noise

Error mitigation (today) ➔ Fight noise

Approaches against quantum noise

We focus on **how noise affects computation (errors)**

Noise sources:

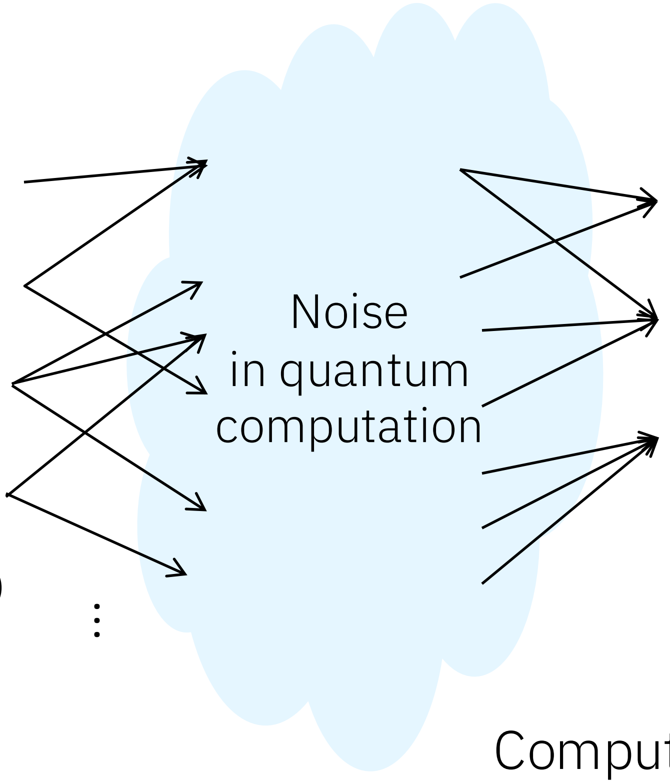
Pulse distortion

Material defects

Heating

Cosmic ray

etc. (innumerable)



Classification of errors
(CS view)

Affected operations:

- **Coherent error** — Gates
- **Incoherent error** — Gates
- **SPAM error** — Resets & measurements
  (SPAM: State Preparation And Measurement)
- **Sampling error** — Measurements

Physics approach:

- Mechanism noise is produced
- How to protect from the noise

Computer science (CS) approach: **Today**

- Effects of noises in computation
- How to minimize the effect of noises (errors)

Coherent error (Unitary error)

[Sources]
- Miscalibration (e.g. pulse amplitudes, qubit frequency)
- Unwanted interaction between qubits

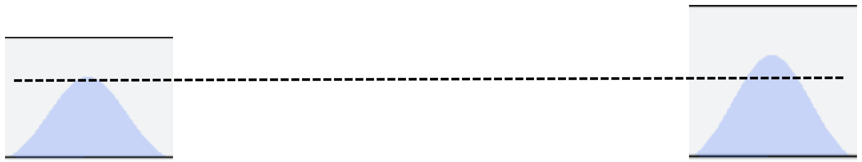
[Characters]
- Unitary evolution, No change in purity (pure state → pure state)

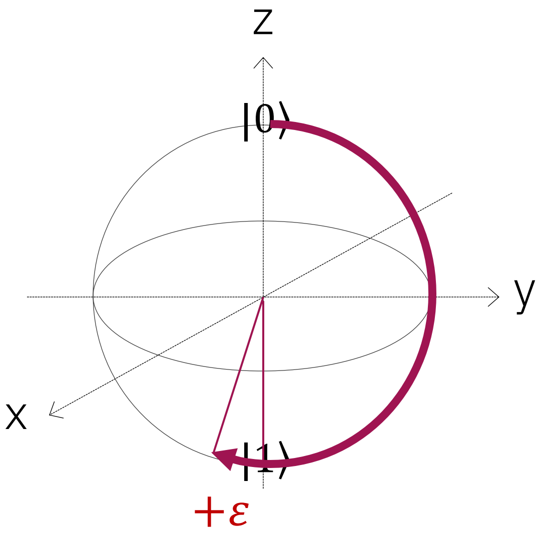
[Measures]
- (Better calibration), Error mitigation/suppression

Ex) Miscalibration of X gate

Miscalibration

$$R_x(\pi) \longrightarrow R_x(\pi + \varepsilon)$$





Purity: $\mathrm{tr}(\rho^2)$

$$\frac{1}{d} \leq \mathrm{tr}(\rho^2) \leq 1$$

Completely mixed — pure

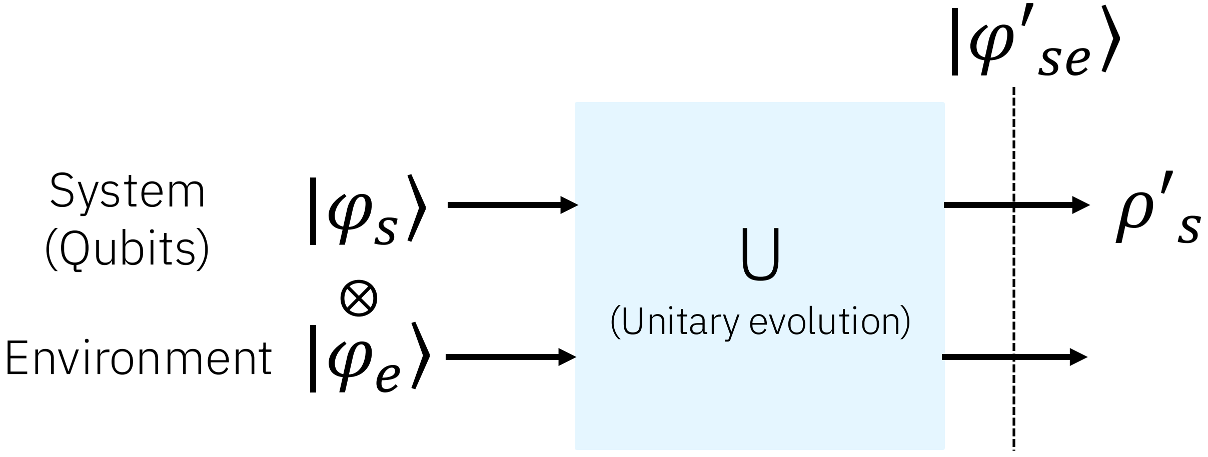
Miscalibration may cause over/under rotation errors

Incoherent error

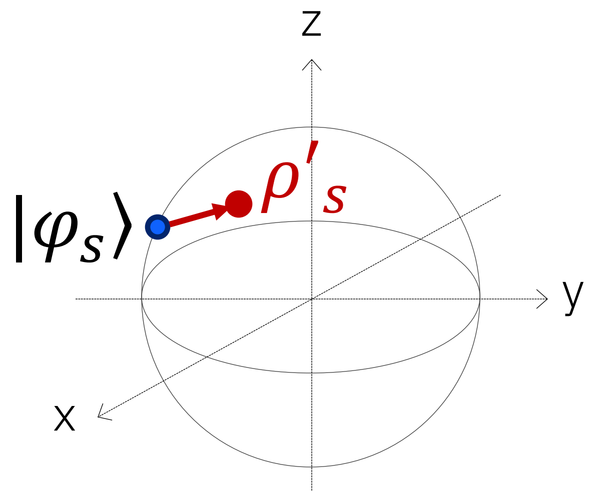
[Sources] • Entanglement (coupling) with environment (system is open)

[Characters] • Non-unitary, Loss of purity (pure state → mixed state)

[Measures] • Error mitigation



Quantum interaction
as a whole



$$\rho'_s = \mathrm{tr}_e(|\varphi'_{se}\rangle\langle\varphi'_{se}|)$$

Partial trace over e (discard environment e, leave system s)

(https://en.wikipedia.org/wiki/Partial_trace)

Extreme example: Subsystem of the Bell state

The Bell state is a pure state, but the reduced density operator of the first qubit is a mixed state (the completely mixed state)

Stronger entanglement (with env.) → More error (on the system)

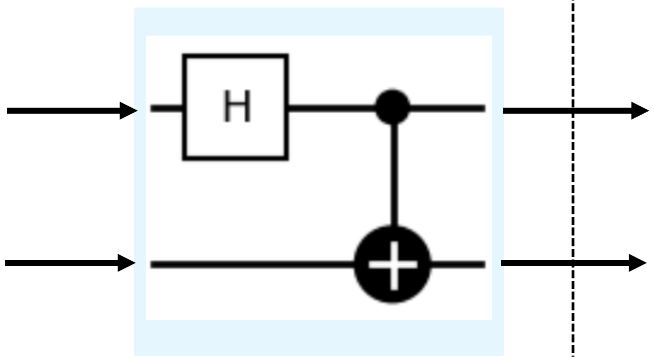
Bell state

$$|\varphi'_{se}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

System (Qubit) $|0\rangle$

$\otimes$

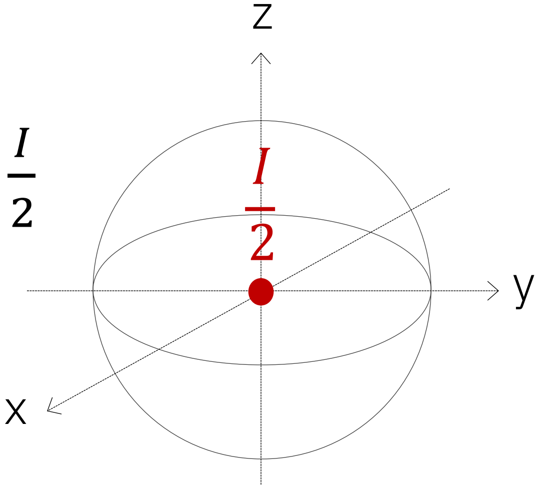
Environment (Qubit) $|0\rangle$



Quantum interaction as a whole

$$\rho'_s = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{I}{2}$$

Completely mixed state



$$\rho'_s = \mathrm{tr}_e(|\varphi'_{se}\rangle\langle\varphi'_{se}|)$$

Partial trace over e (discard environment e, leave system s)

2. SPAM error and Sampling error

SPAM error is state preparation error and measurement error. Sampling error (shot error) comes from the nature of physics. And, from the graph of circuit depth and observable Z measurement results, you will learn what effect different types of noise have.

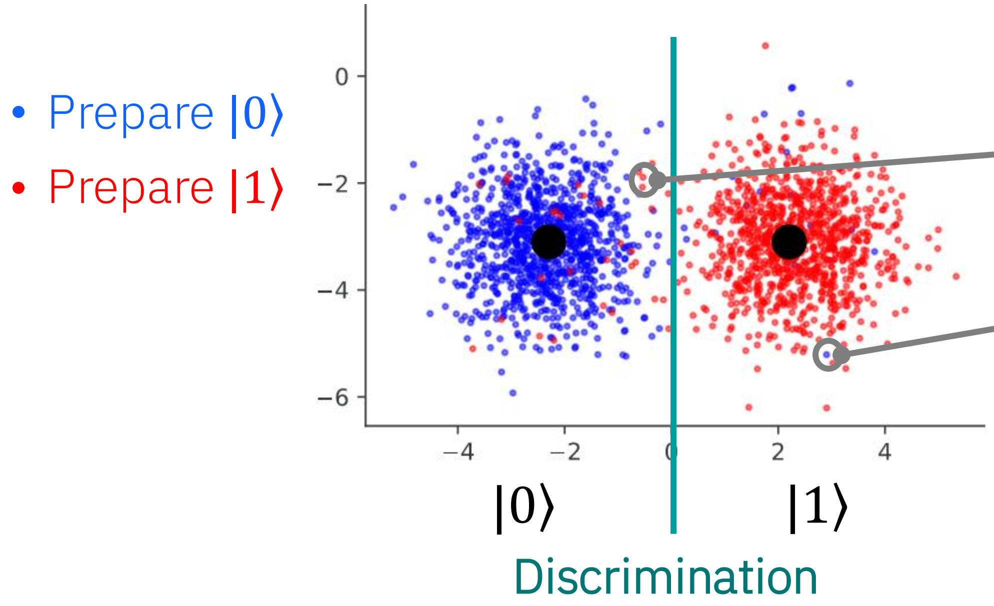
Measurement or Readout error (SPAM error) (SPAM: State Preparation And **Measurement**)

[Sources] • Mis-discrimination (in qubit state readout)

[Characters] • Classical errors (bit-flip errors)

[Measures] • (Better calibration), Error mitigation

Ex) IQ plot for qubit state readout



Two types of readout errors

$P(0|1)$: Prepare $|1\rangle$, Measure $|0\rangle$

$P(1|0)$: Prepare $|0\rangle$, Measure $|1\rangle$

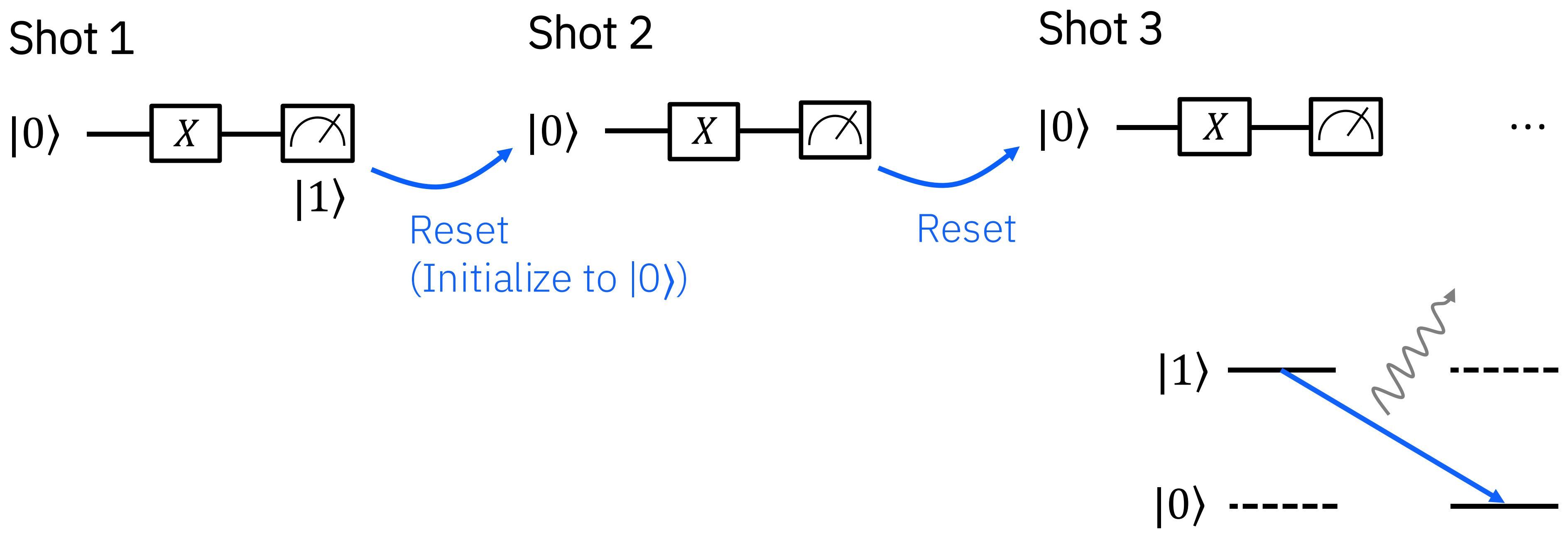
Initialization or Reset error (SPAM error)

(SPAM: **State Preparation** And Measurement)

[Sources] • Imperfect reset (of previously measured state)

[Measures] • Long shot intervals

Shots: Run a circuit multiple times to sample results (bits)



Sampling error (Shot error)

[Sources] • Core nature of quantum physics

[Measures] • Increase the number of shots

Measure a qubit → Observe a bit 0 or 1, following

Bernoulli distribution

$$\begin{cases} 0 \text{ with probability } p \\ 1 \text{ with probability } 1-p \end{cases}$$

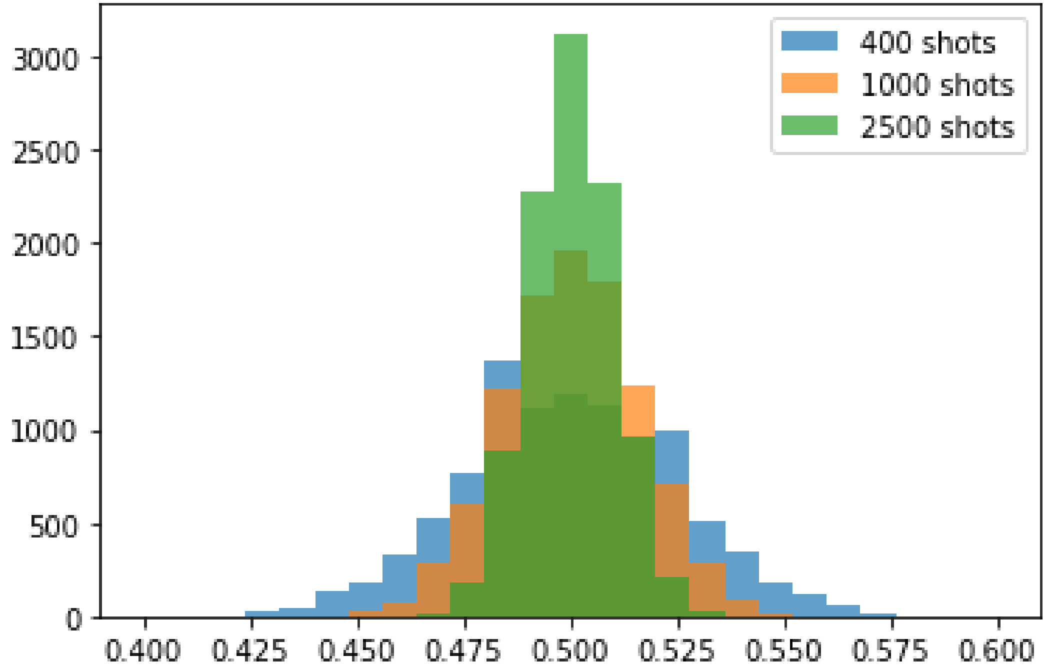
p depends on amplitude of $|0\rangle$ of the state

Measure multiple times to know p

→ Obtain sample mean of Bernoulli random variables $\hat{p}$

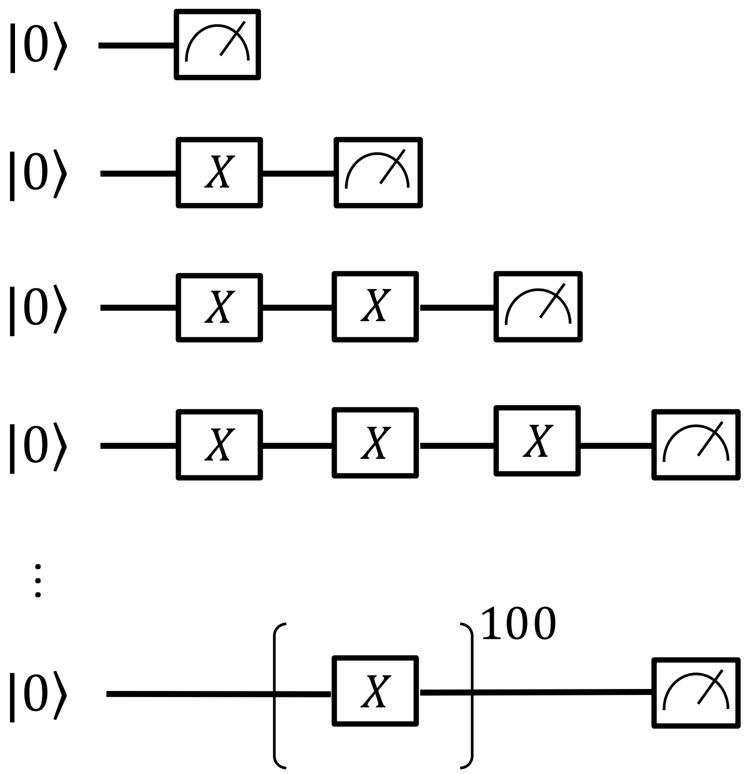
More shots ➔ Less variance (more precise $\hat{p}$)

Distribution of mean of Bernoulli random variables (p=0.5)



Quiz: What errors look like

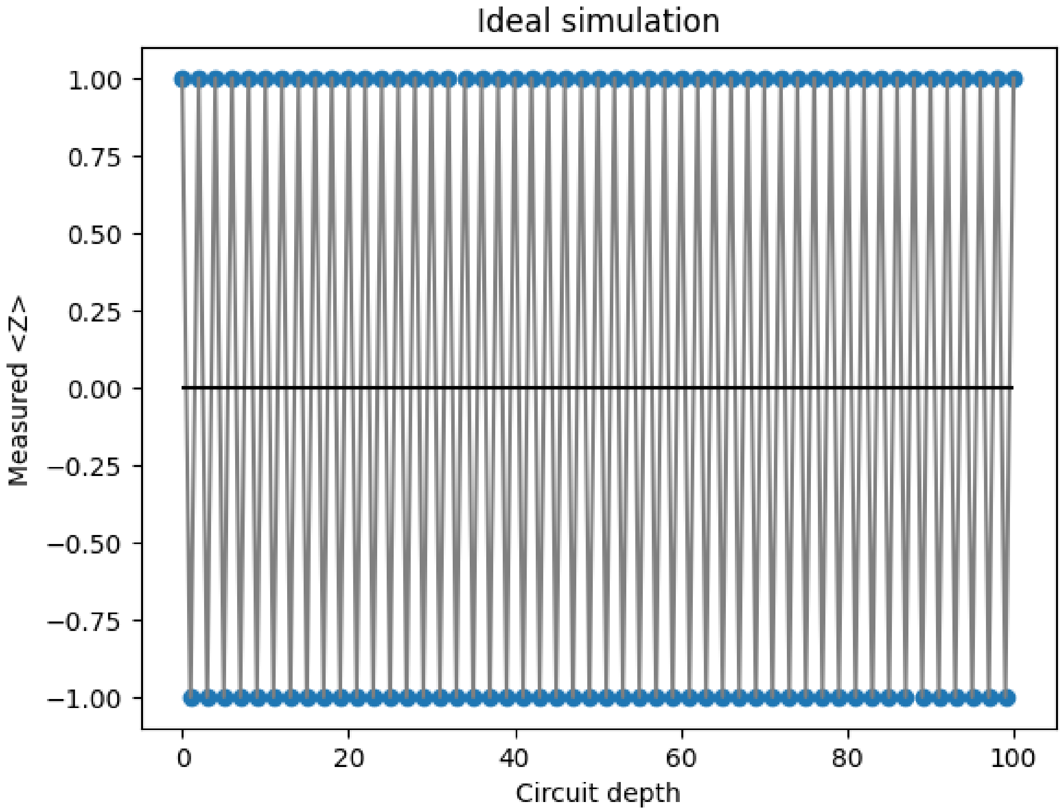
Run the following 101 circuits



400 shots for each circuit

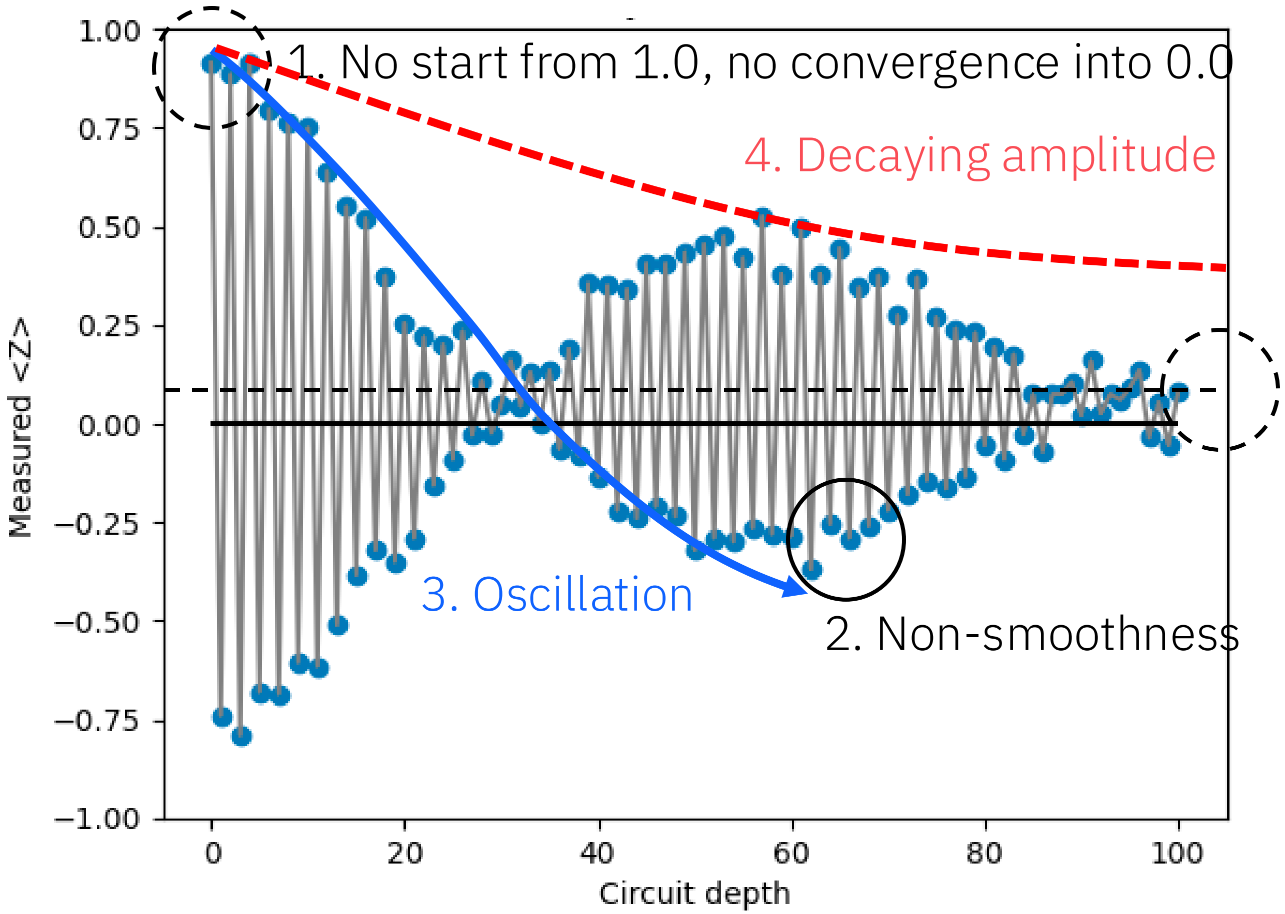
Plot $\langle Z\rangle = \langle\varphi|Z|\varphi\rangle = P(0) - P(1)$

Ideally, observe 1 and -1 alternatively



Quiz: What errors look like

Running on noisy quantum computer, we observe

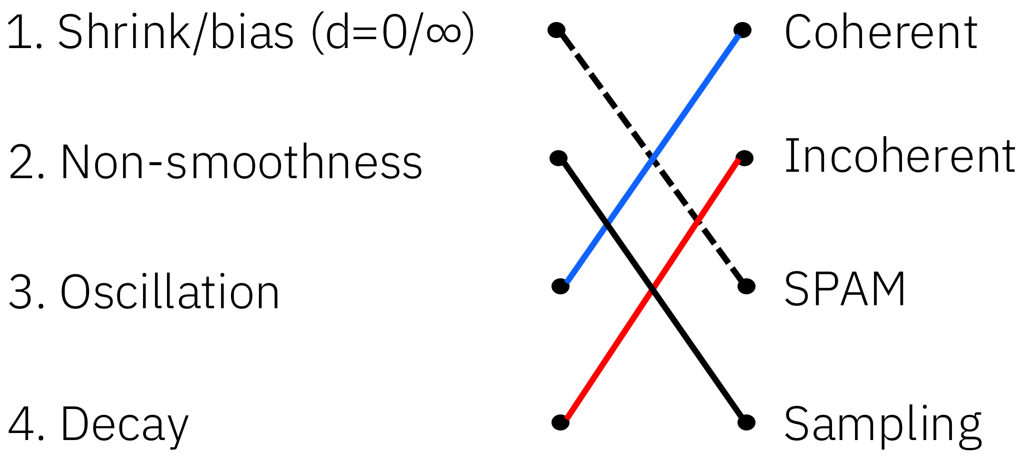


Why?

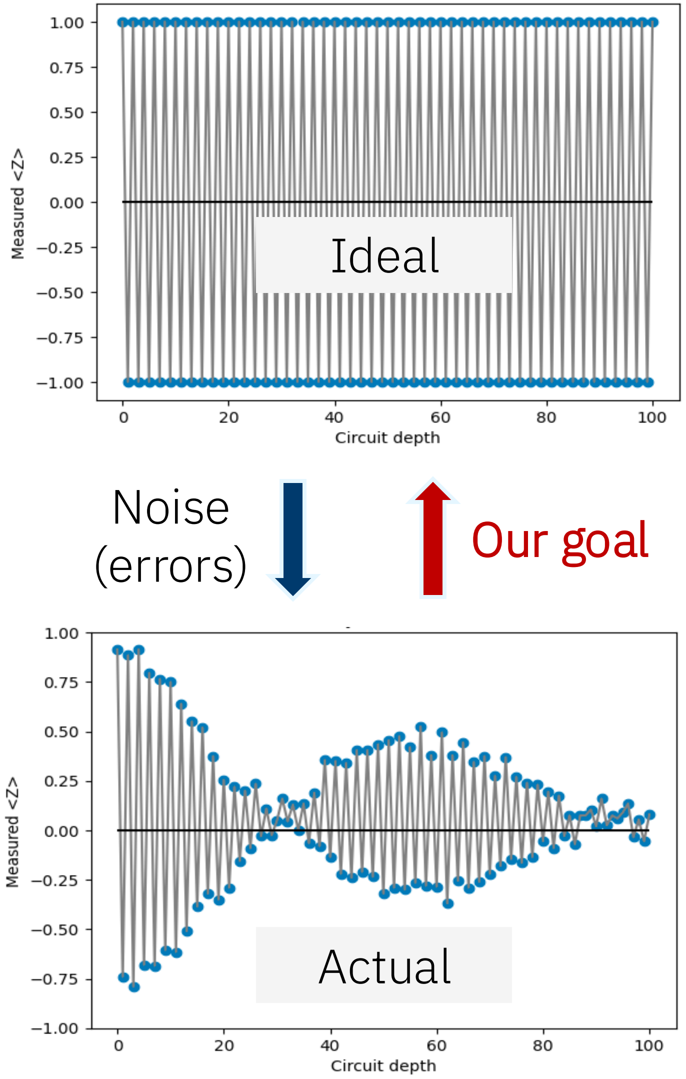
Connect an observation with the error causing it by a line

Observations:

Errors:



Fight gate/measurement errors



- Gate and measurement errors are dominant in today's superconducting-qubit computers

Classification of errors (CS view)	Affected operations:
• Coherent error	Gates
• Incoherent error	Gates
• SPAM error	Resets & measurements
• Sampling error	Measurements

(SPAM: State Preparation And Measurement)

3. Error suppression and mitigation

Error suppression is a technique aiming to reduce the error itself during the execution of a circuit. In contrast, error mitigation is a technique aiming to recover the error-free result using classical post-processing. You will learn an example of error suppression: Dynamical Decoupling (DD).

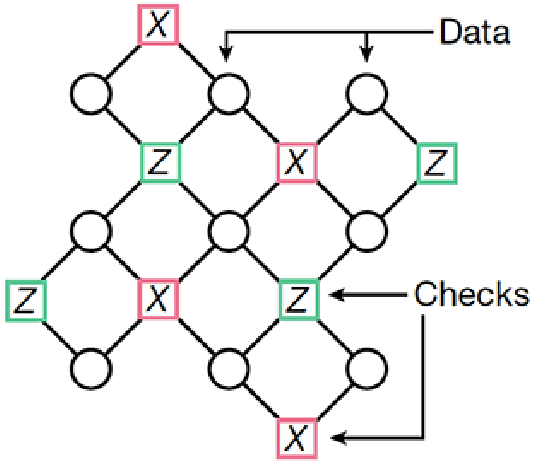
What you learn today

- Talk (30min)
  - What is quantum noise/error
  - **Error suppression and mitigation techniques**
    - TREX (Twirled Readout Error eXtinction)
    - ZNE (Zero Noise Extrapolation)
    - PEA (Probabilistic Error Amplification)
- Break
- Hands-on (20 min)
- Theory (Hard – 30min)
  - Formalism of quantum errors
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
    - PTM (Pauli Transfer Matrix) representation

Error correction or error mitigation?

How to deal with errors due to noise?

Quantum error correction (QEC)

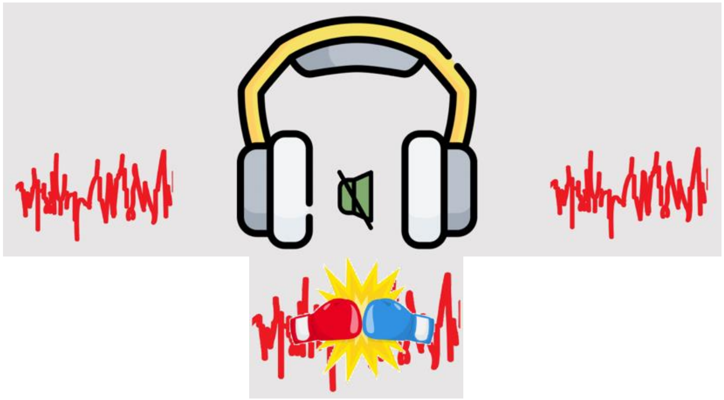


Source: Fig. 1 in [1]

Monitor
Error occurs
Error detected

Correct in quantum computation (in real time)

Quantum error mitigation (QEM)



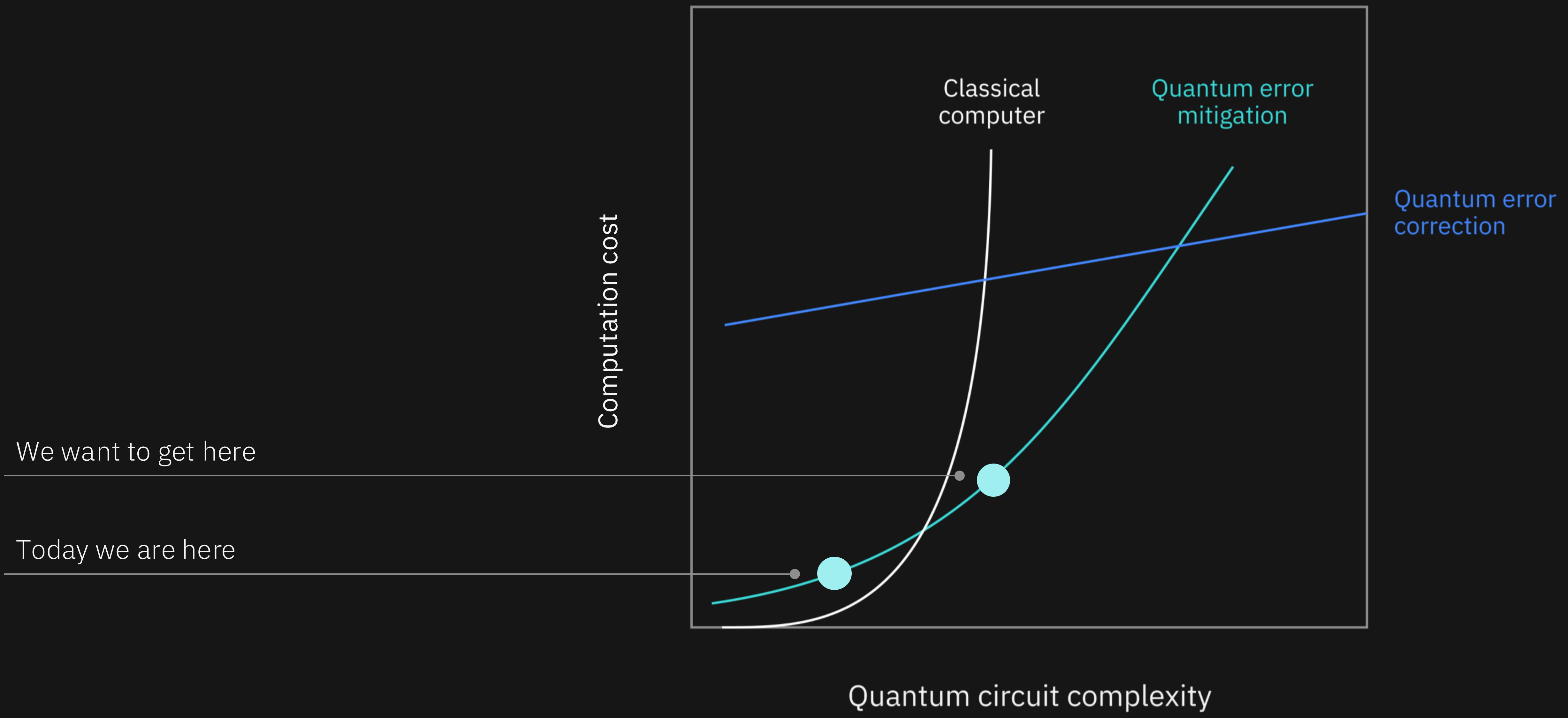
Source: [2]

No monitor
Error occurs
Error undetected

Estimate corrected with classical computation (by post processing)

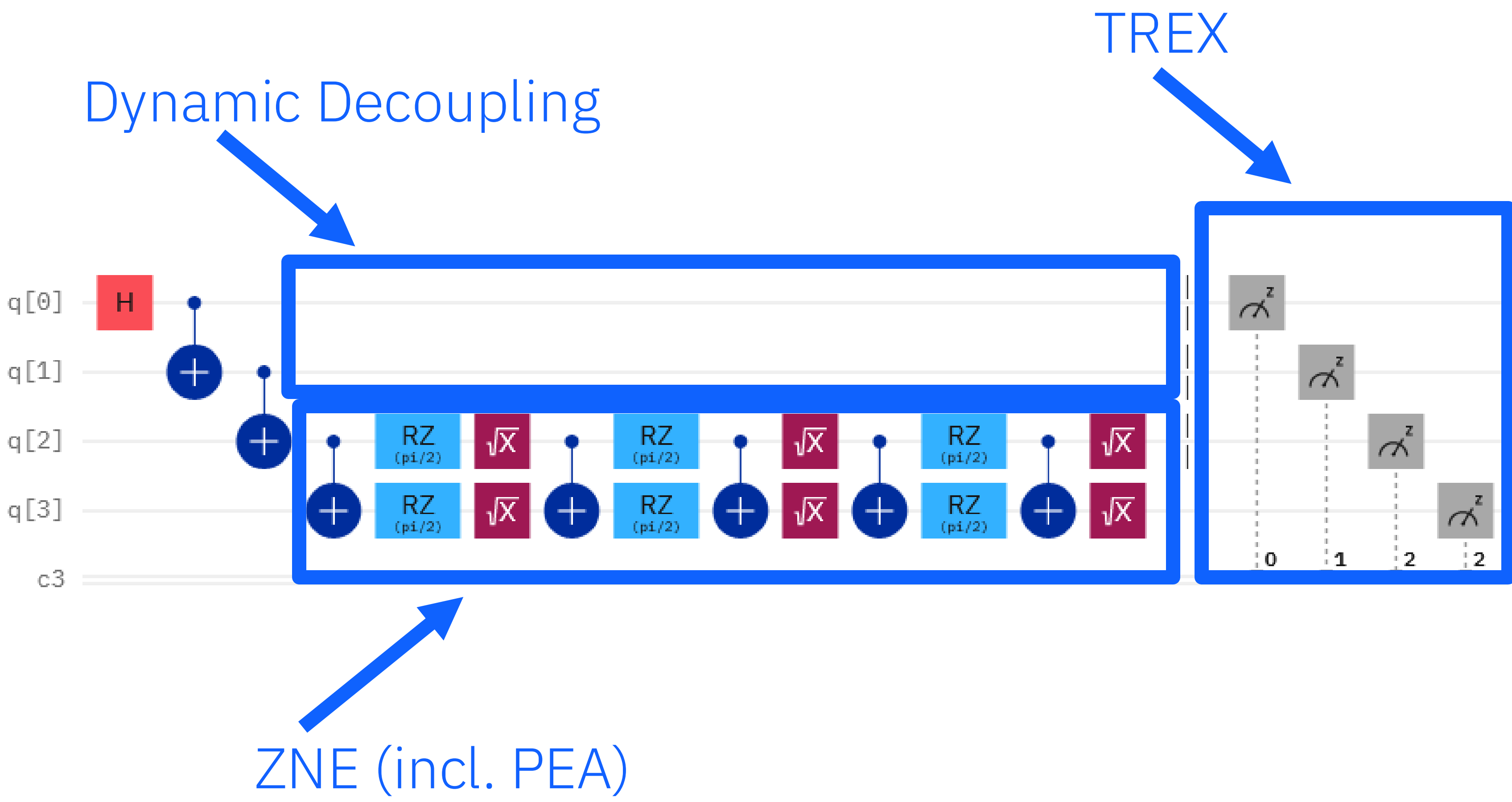
[1] Bravyi, S., Cross, A.W., Gambetta, J.M. *et al.* High-threshold and low-overhead fault-tolerant quantum memory. *Nature* **627**, 778–782 (2024).
[2] Minov Z., Probabilistic Error Cancellation with Sparse Pauli-Lindblad Models on Noisy Quantum Processors (https://www.youtube.com/watch?v=oPSBivh2rxQ)

Quantum Error Mitigation and Correction



Error suppression and mitigation techniques

- Different types of errors need different suppression and mitigation techniques.
- Different types of techniques can be combined!



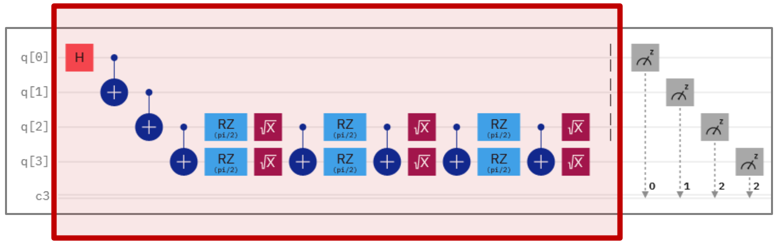
- TREX (Twirled Readout Error eXtinction)
- ZNE (Zero Noise Extrapolation)
- PEA (Probabilistic Error Amplification)

Error suppression and Error mitigation

Error suppression

Aim to reduce the error itself (in real time)

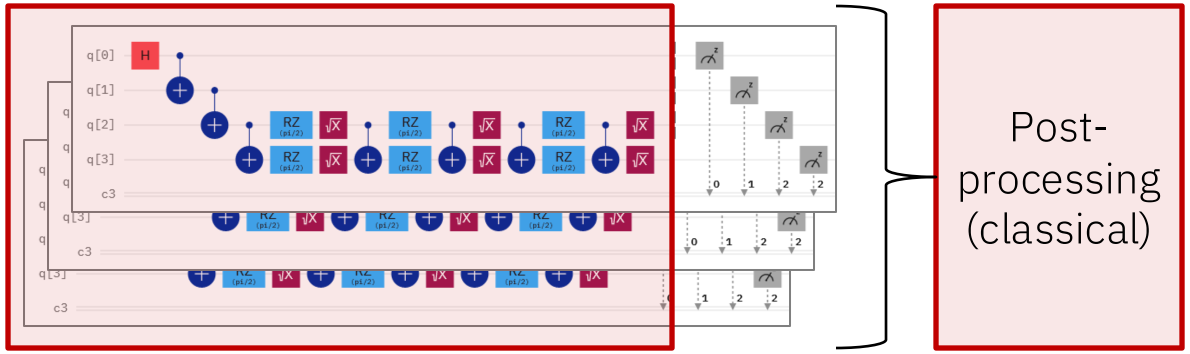
- Do something before measurement
- No change in the number of circuits
- Work even for a single shot

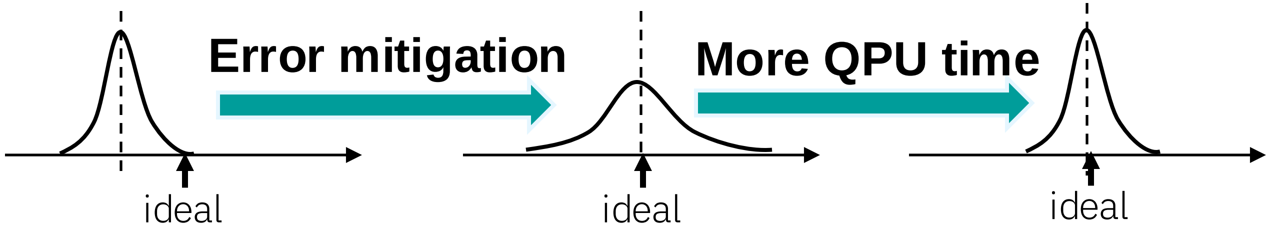


Error mitigation

Aim to recover the error-free result (with post-processing)

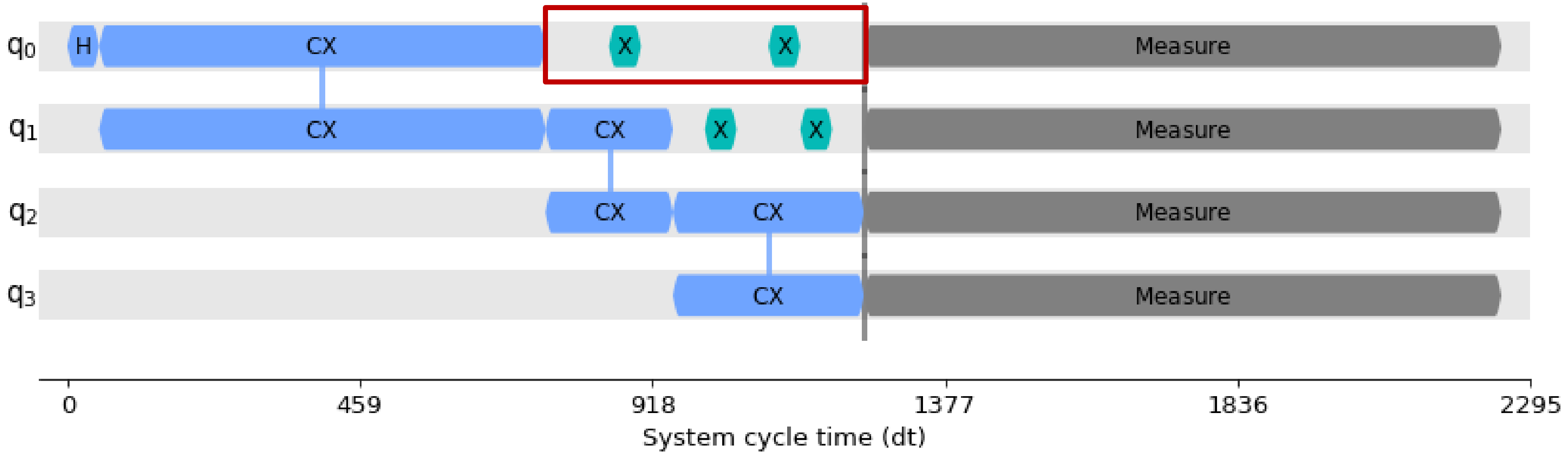
- Require classical post-processing
- Require more circuits to run
- Require multiple shots





Error suppression: Dynamical Decoupling (DD)

- **Suppress errors in qubit idling time** effectively
- Insert gates add up to the identity, e.g. X—X, X—Y—X—Y



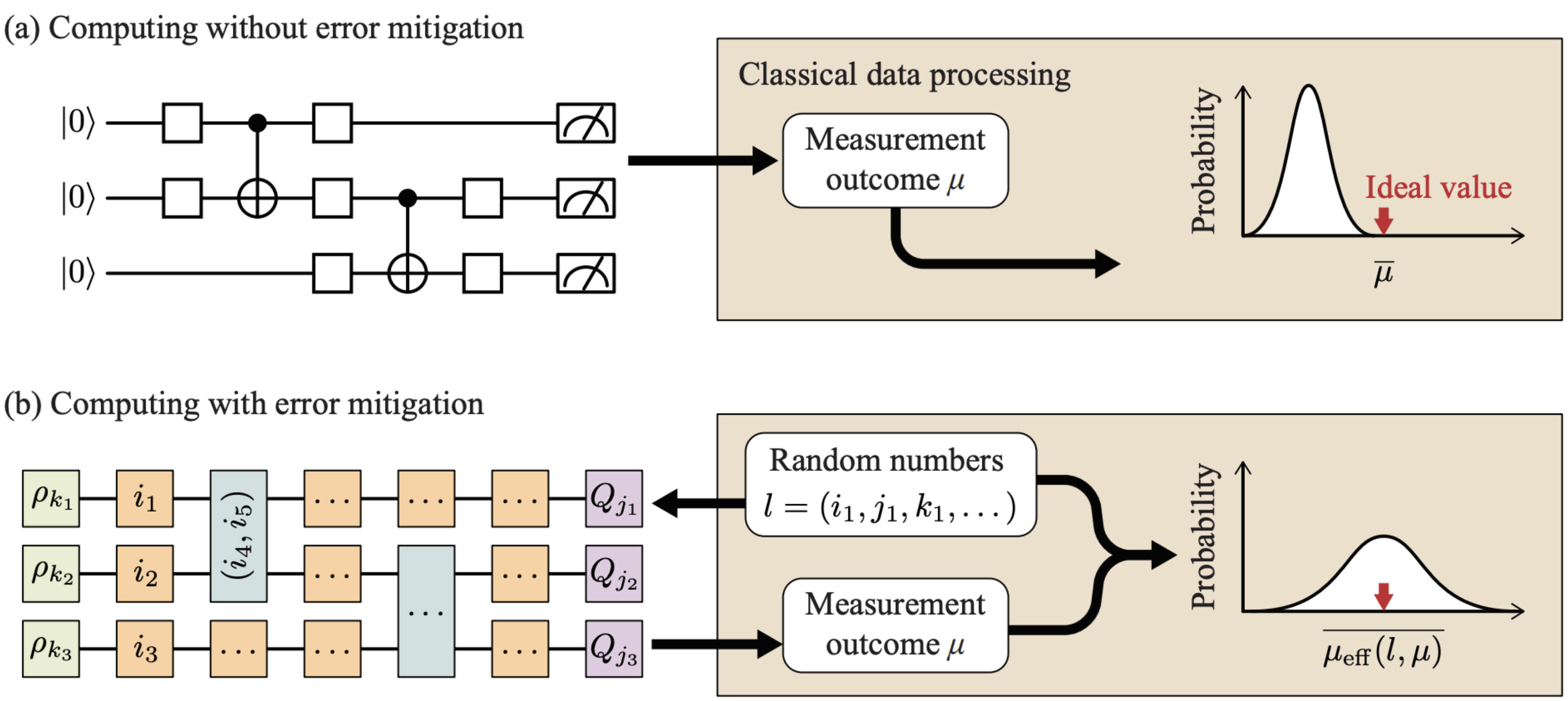
In this case, at least coherent Rz errors are cancelled out (assuming no errors on X gates):

$R_z(\theta)\ X\ R_z(2\theta)\ X\ R_z(\theta) = R_z(\theta)\ R_z(-2\theta)\ R_z(\theta) = I$ No DD → $R_z(4\theta)$ error

Error Mitigation

Obtain a **more accurate expectation value** at the price of larger variance

➔ Need **more sampling** (= more time) to keep variance to the same degree



Source: Fig. 1 in [2]

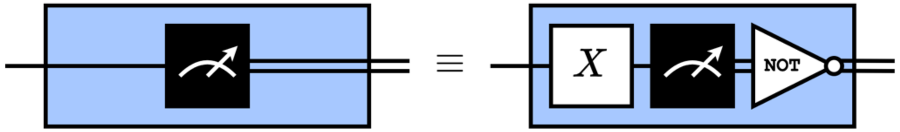
[1] Endo, S., Benjamin, S. C. & Li, Y. Practical quantum error mitigation for near-future applications. Physical Review X 8, 031027 (2018).

4. Error mitigation techniques

There are three major error mitigation techniques:
Twirled Readout Error eXtinction (TREX), Zero Noise Extrapolation (ZNE), and Probabilistic Error Amplification (PEA).

Twirled Readout Error eXtinction (TREX)

EV: Expectation Value

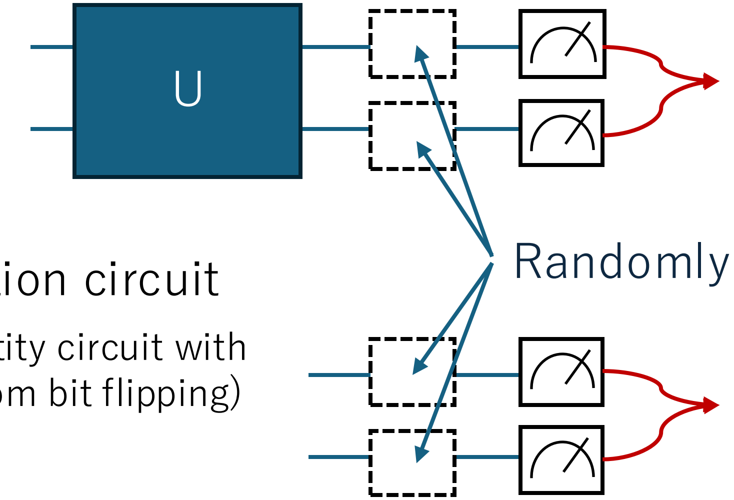
Idea: 

Focus on the computation of the EVs of Pauli observables composed only of Pauli I and Z for simplicity

Ex) EV of ZZ for $|\varphi\rangle = \mathrm{U}|00\rangle$, i.e. $\langle\varphi|ZZ|\varphi\rangle$

$$\langle ZZ\rangle = P(00) - P(01) - P(10) + P(11)$$

1. Original circuit (with random bit flipping)



Flip measured bit if X gate is inserted → $\langle\tilde{\varphi}|ZZ|\tilde{\varphi}\rangle$

Randomly insert X gate

2. Calibration circuit

(Identity circuit with random bit flipping)

Flip measured bit if X gate is inserted → $\langle\tilde{0}|ZZ|\tilde{0}\rangle$

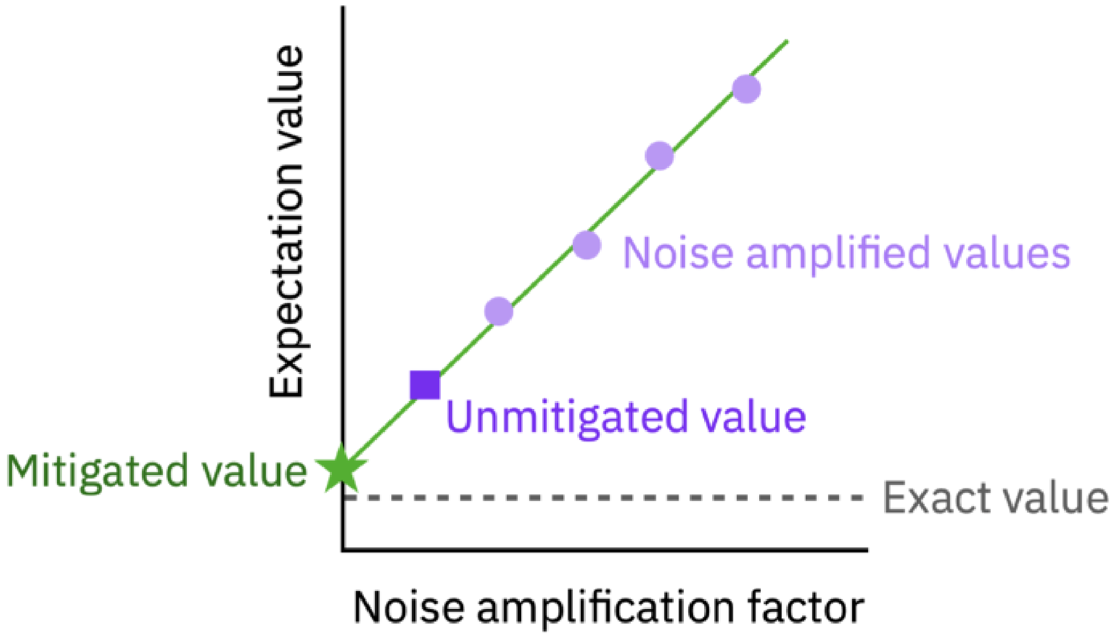
Mitigated EV = [EV from 1] / [EV from 2]

$\langle\tilde{\varphi}|ZZ|\tilde{\varphi}\rangle$ $\langle\tilde{0}|ZZ|\tilde{0}\rangle$

Van Den Berg, E., Minev, Z. K., & Temme, K. (2022). Model-free readout-error mitigation for quantum expectation values. *Physical Review A*, 105(3), 032620.

Zero Noise Extrapolation (ZNE)

Run multiple circuits with different gate error rates and extrapolate the expectation value at zero-noise point



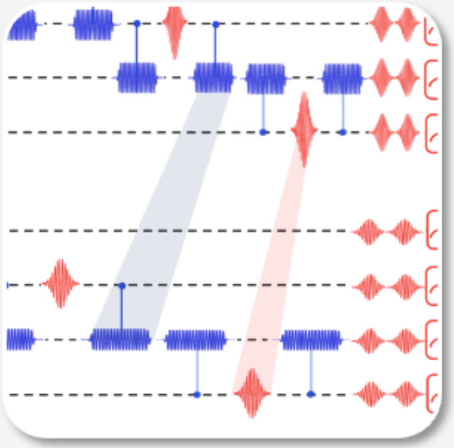
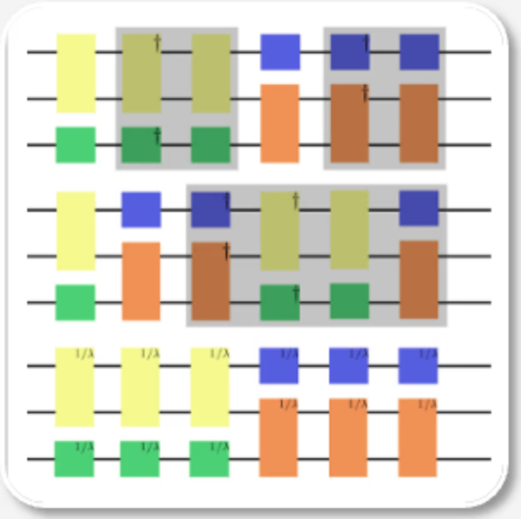
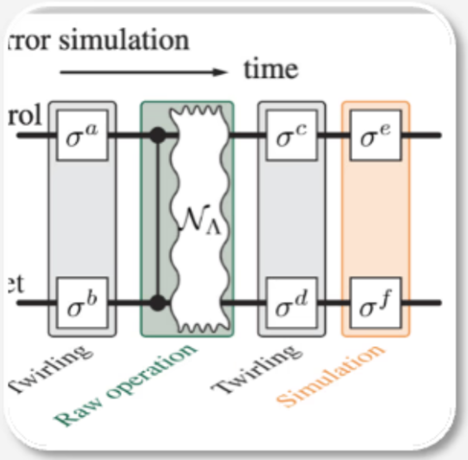
Options:

- Noise amplifier
- Noise factors — e.g. [1.0, 1.2, 1.5], [1, 3, 5]...
- Extrapolator — e.g. Linear, Quadratic, Exponential ...

https://github.com/Qiskit/qiskit-ibm-runtime/blob/stable/0.17/docs/tutorials/Error-Suppression-and-Error-Mitigation.ipynb

ZNE: Noise Amplification

- Pulse stretching assumes gate noise is proportional to duration, which is typically not true. Calibration is also costly.
- Gate folding requires large stretch factors that greatly limit the depth of circuits that can be run.
- PEA can be applied to any circuit that can be run with native noise factor (λ=1λ=1) but requires learning the noise model.
- You can write your own amplification!

Pulse Stretching	**Gate Folding**	**Probabilistic Error Amplification**
Scale pulse duration via calibration	Repeat gates in identity cycles $U \mapsto U(U^{-1}U)^{\lambda-1}/2$	Add noise via sampling Pauli channels
		
Kandala et al. Nature (2019)	Shultz et al. PRA (2022)	Li & Benjamin PRX (2017)

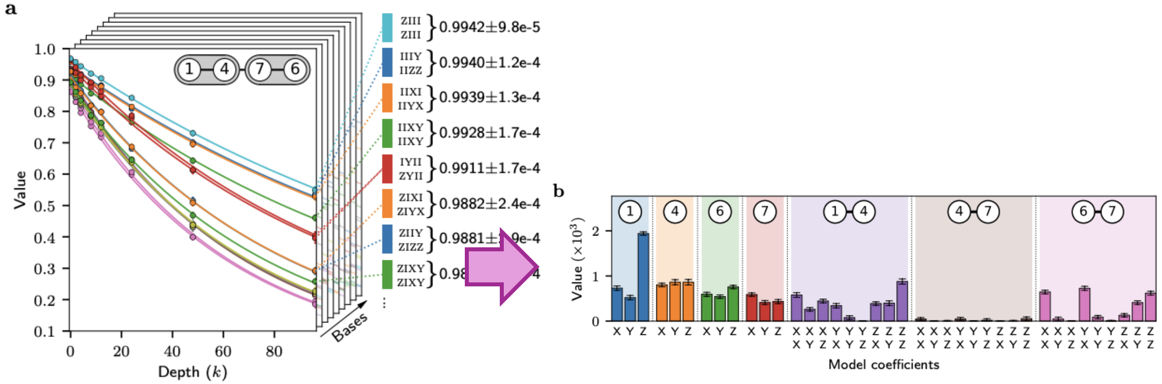
Probabilistic Error Amplification (PEA)

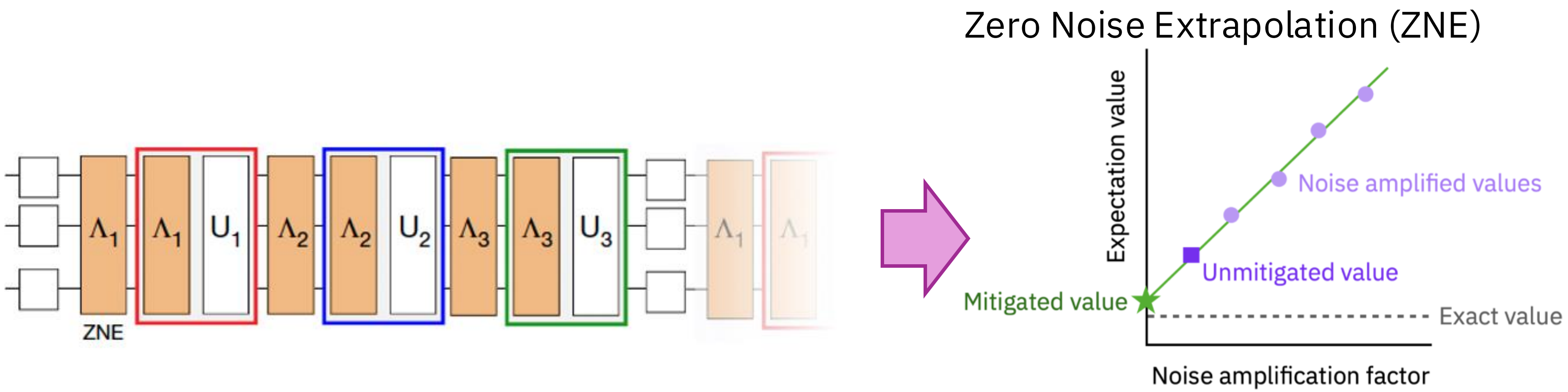
Pauli Twirling

1) Simplify noise: Gate noise → Pauli channel

2) Learn noise (Estimate Pauli channel params)

3) Amplify noise + ZNE





Kim, Y., Eddins, A., Anand, S., Wei, K. X., Van Den Berg, E., Rosenblatt, S., ... & Kandala, A. (2023). Evidence for the utility of quantum computing before fault tolerance. *Nature*, *618*(7965), 500-505.

Pauli Twirling

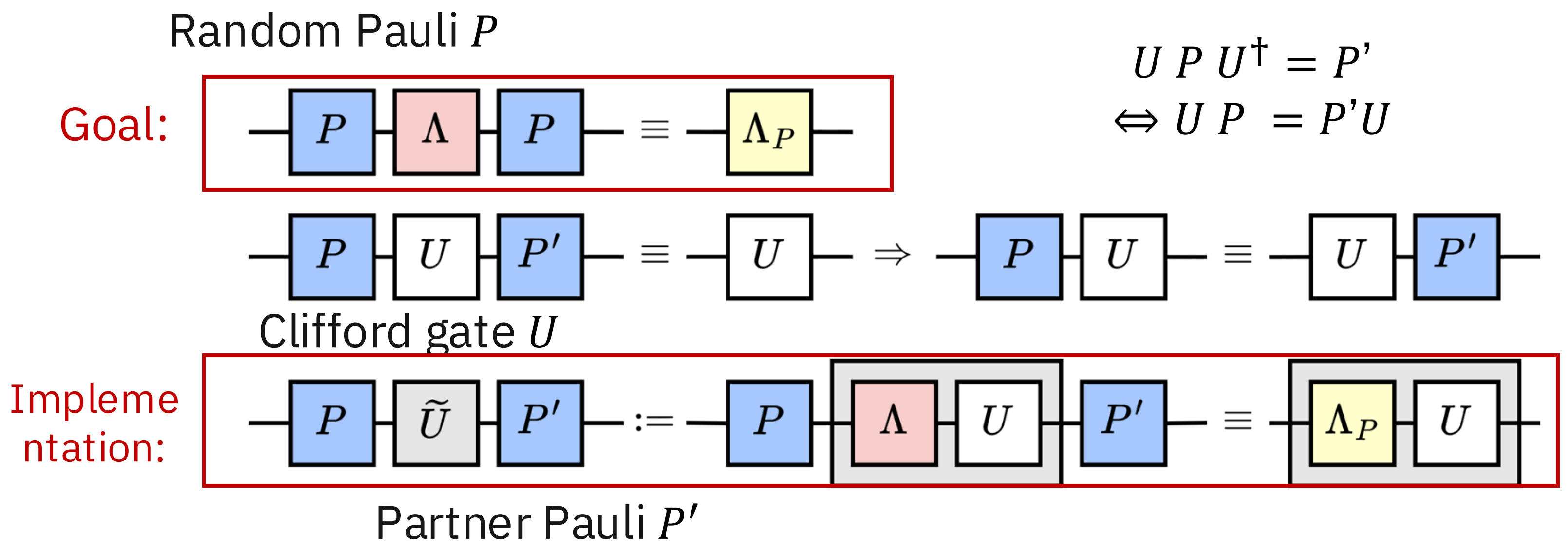
- Also called **randomized compiling**.
- Used to convert arbitrary noise channels into **Pauli channels**.
- Helps when dealing with coherent noise.
- Helps in the extrapolation stage of ZNE by making noise increase more or less monotonically.
- Often exclusively used on two qubit gates.

Clifford group | Pauli group

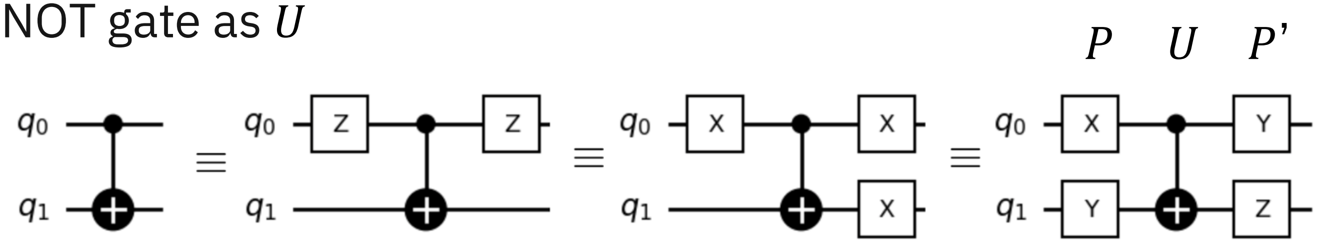
$$\mathbf{C}_n = \{V \in U_{2^n} \mid V\mathbf{P}_n V^\dagger = \mathbf{P}_n\}$$

Clifford maps a Pauli to another Pauli by conjugation

$$U\,P\,U^\dagger = P'$$
$$\Leftrightarrow U\,P = P'U$$



Ex) CNOT gate as U



7. Formalism of quantum errors

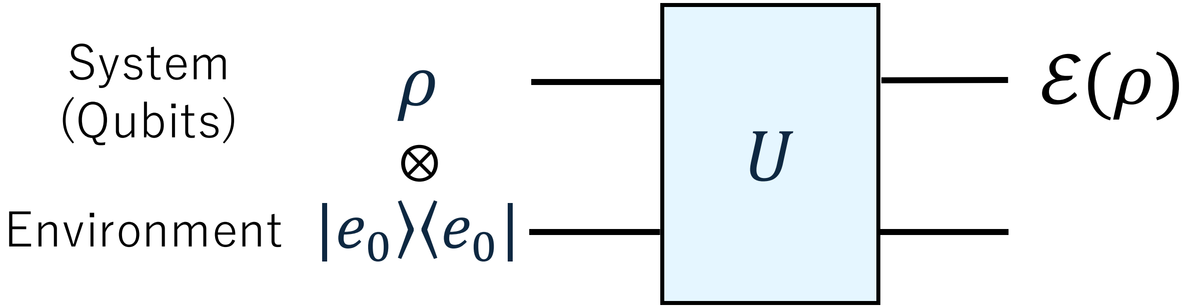
We can describe the quantum error as a quantum channel which is a linear map of a quantum state to another quantum state. You will learn the standard quantum error channels and Pauli channel.

What you learn today

- Talk (35min)
  - What is quantum noise/error
  - Error suppression and mitigation techniques
    - TREX (Twirled Readout Error eXtinction)
    - ZNE (Zero Noise Extrapolation)
    - PEA (Probabilistic Error Amplification)
- Break
- Hands-on (20 min)
- **Theory (30min – Hard)**
  - **Formalism of quantum errors**
    - Standard error channels, e.g. Pauli error channel
    - Quantum channel
    - PTM (Pauli Transfer Matrix) representation

System-Environment representation of noise

- (Incoherent) error is from entanglement with environment

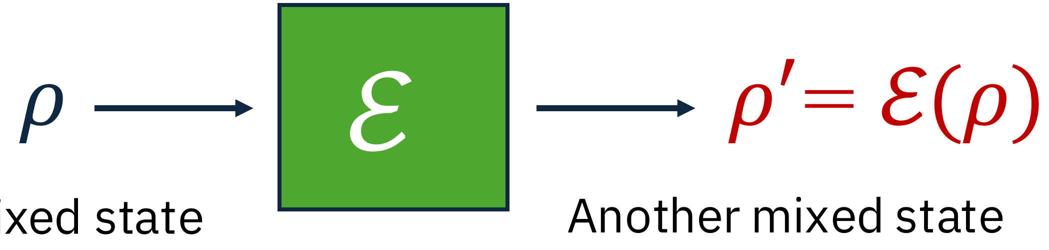


- Any quantum error on the system is fully characterized by U
  - Include coherent (unitary) error as a special case $U = U_s \otimes U_e$
- Difficult to describe the environment explicitly
  - Difficult to know U directly

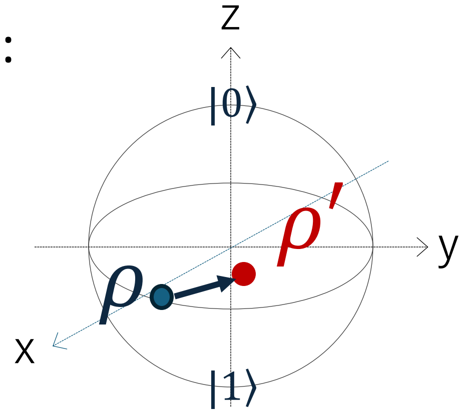
Quantum Channel

A linear map of a **quantum state** to another **quantum state**

Quantum Channel



1-qubit case:



Recall we use density matrix to represent a mixed state

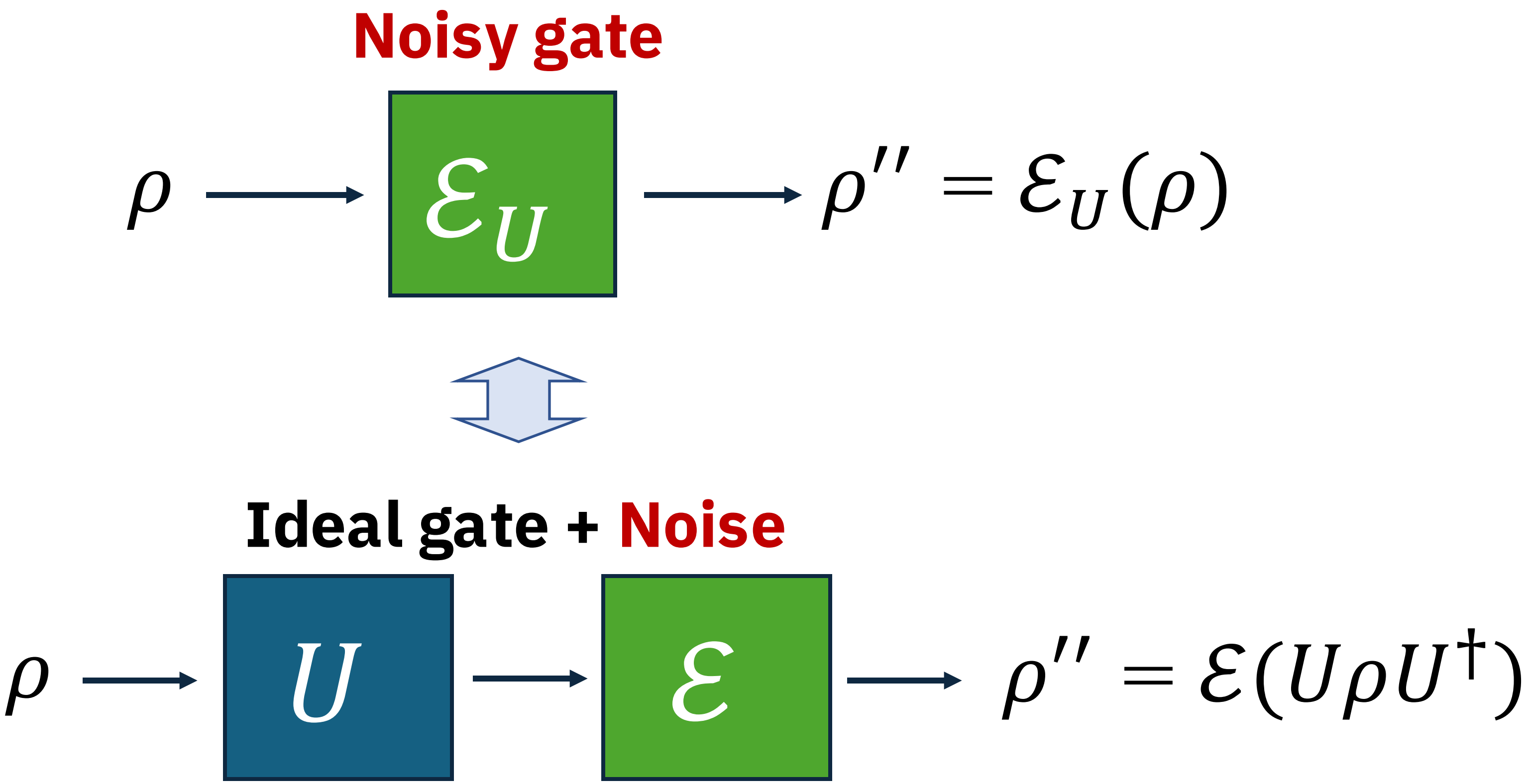
$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|$$

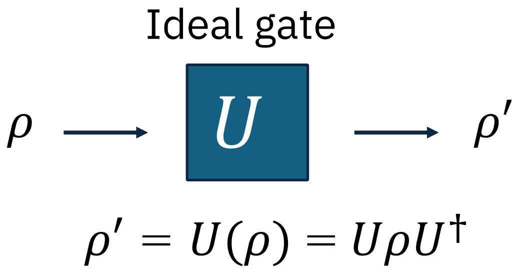
$$= \frac{1}{2}(\mathrm{I} + r_x X + r_y Y + r_z Z)$$

N-qubit density matrix is 2^N by 2^N

Noisy gate is a quantum channel

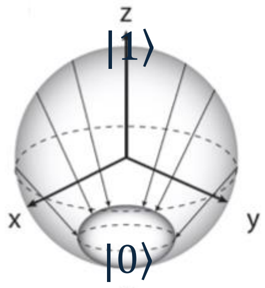
Two ways to represent a noisy gate using quantum channel





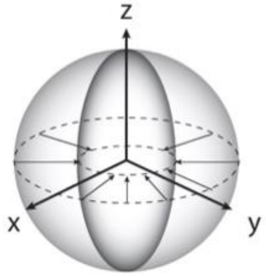
Common quantum errors

Incoherent errors



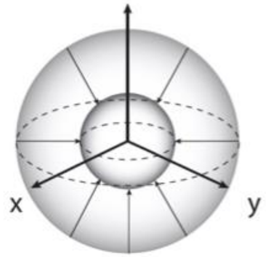
Amplitude damping error:

Relaxation error(|1> → |0>)



Phase damping error (dephasing):

Loss of phase information



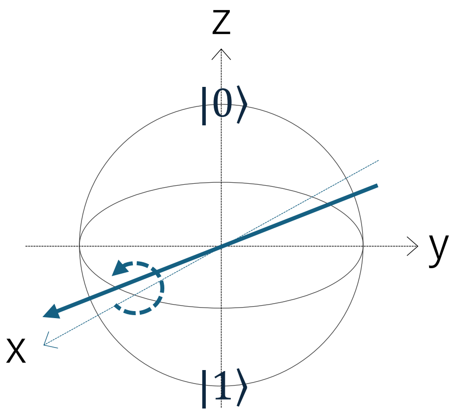
Depolarizing error:

Isotropic loss of purity (A special case of Pauli error)

Pauli error:

Different loss in X/Y/Z direction (see the next page for the details)

Coherent errors



Unitary error:

Miscalibration (over-/under-rotation)

$$\rho \mapsto U\rho U^{\dagger}$$

Figures cited from Quantum benchmarking - Petar Jurcevic - QCHS 2022 (https://www.youtube.com/watch?v=rM0iOVKqUFE)

Pauli error (channel)

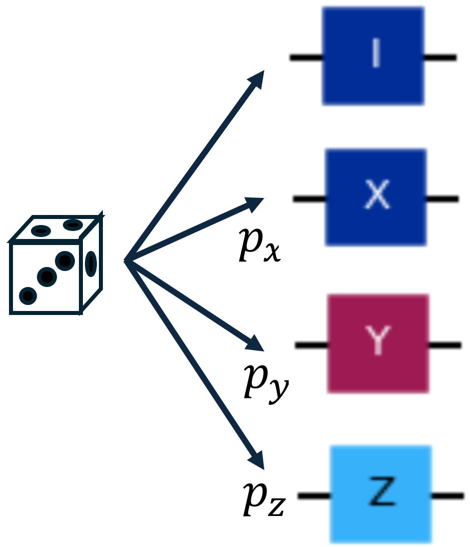
1q-Paulis:

$$\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Random application of Pauli gates

1-qubit case:

- Apply X with probability p_x,
- Apply Y with probability p_y,
- Apply Z with probability p_z,
- Apply I with probability $1 - p_x - p_y - p_z$



2-qubit case:

- Apply XI, XX, XY, XZ with probability $p_{XI}, p_{XX}, p_{XY}, p_{XZ}$
- Apply YI, YX, YY, YZ with probability $p_{YI}, p_{YX}, p_{YY}, p_{YZ}$
- Apply ZI, ZX, ZY, ZZ with probability $p_{ZI}, p_{ZX}, p_{ZY}, p_{ZZ}$
- Apply IX, IY, IZ with probability p_{IX}, p_{IY}, p_{IZ}
- Apply II with the rest probability $1 - p_{IX} - p_{IY} - p_{IZ} - \cdots$

N-qubit case: 4^N random Pauli application (with $4^N - 1$ parameters)

(Abbreviation for Pauli XY $= X \otimes Y$)

All $4^N - 1$ parameters are the same ➔ depolarizing error

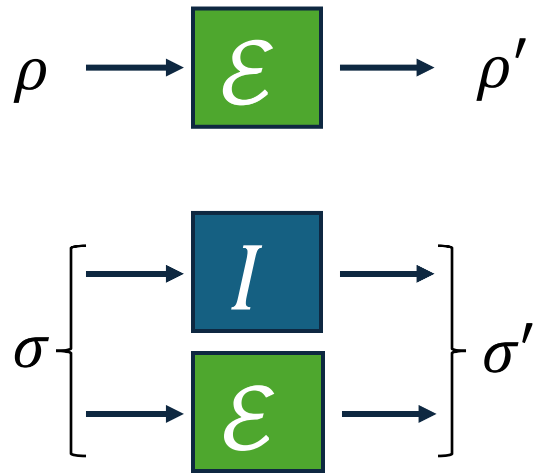
8. Theory of quantum channel

Quantum channel is mathematically a CPTP map: CP stands for completely positive, TP stands for trance preserving. And any quantum channel (CPTP map) has a Kraus representation.

[Theory] Quantum Channel := CPTP-map

Properties required for a mapping $\mathcal{E}$ between quantum states:

1. CP (Completely Positive): $\rho \succeq 0 \Rightarrow \mathcal{E}(\rho) \succeq 0 \quad \text{and} \quad \sigma \succeq 0 \Rightarrow (I \otimes \mathcal{E})(\sigma) \succeq 0$
2. TP (Trace Preserving): $\mathrm{tr}(\mathcal{E}(\rho)) = \mathrm{tr}(\rho) = 1$
3. Convex linear: $\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$



⇕ [Choi 1975]

$\mathcal{E}$ has a Kraus representation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

[Choi 1975] M. D. Choi. Completely Positive Linear Maps on Complex Matrices. Linear Algebrra Appl., 10(285), 1975.

Kraus (Operator-Sum) representation

Any quantum channel can be represented by

$$\text{a set of operators (matrices) } \{E_k\} \text{ s.t. } \sum_k E_k^\dagger E_k = I$$

$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

$$\mathcal{E}^{\text{Kraus}} : \rho \mapsto \sum_k E_k \rho E_k^\dagger$$

$\Leftrightarrow$

(Physical interpretation)

Randomly take one of k states:

$\rho_k = \dfrac{E_k \rho E_k^\dagger}{\mathrm{tr}(E_k \rho E_k^\dagger)}$ with probability $\mathrm{tr}(E_k \rho E_k^\dagger)$

Ex. 1) Gate / Unitary evolution (special case: |k|=1)

$$\rho \mapsto U\,\rho\,U^\dagger \qquad\qquad U^\dagger U = I$$

Examples: Kraus (Operator-Sum) representation

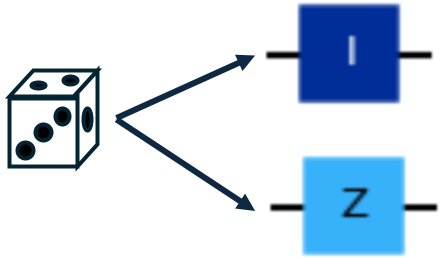
$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

Ex. 2) Positive operator-valued measurement (POVM)

Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex. 3) (Probabilistic) Mixture of unitaries



50% Pauli I - 50% Pauli Z

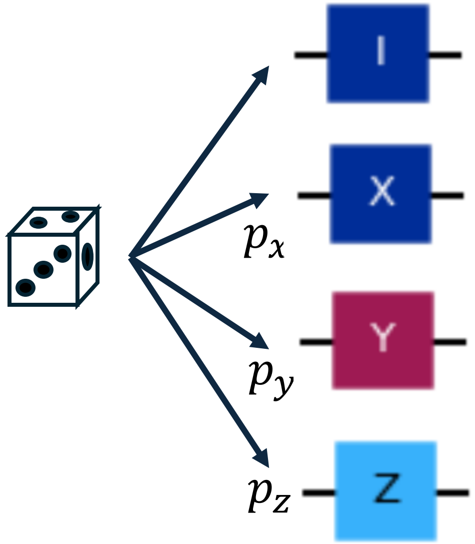
$$E_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise: Kraus representation of 1q Pauli error?

Pauli error = Random application of Pauli gates

Ex) 1-qubit case

- Apply X with probability p_x,
- Apply Y with probability p_y,
- Apply Z with probability p_z,
- Apply I with probability $1 - p_x - p_y - p_z$



$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

Describe Kraus operators representing 1q Pauli error above.

Answer: $\sqrt{p_x}\,X, \sqrt{p_y}\,Y, \sqrt{p_z}\,Z, \sqrt{1 - p_x - p_y - p_z}\,I$

Physical interpretation of Kraus operators

Depends on E_k and ρ!!

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

With probability $\mathrm{tr}(E_k \rho E_k^\dagger)$

transferred into one of k states: $\rho_k = \frac{E_k \rho E_k^\dagger}{\mathrm{tr}(E_k \rho E_k^\dagger)}$

How to process Kraus instructions on state-vector simulator:

Assume $\rho = |\psi\rangle\langle\psi|$

With probability $\mathrm{tr}(E_k|\psi\rangle\langle\psi|E_k^\dagger) = \||E_k|\psi\rangle\||$

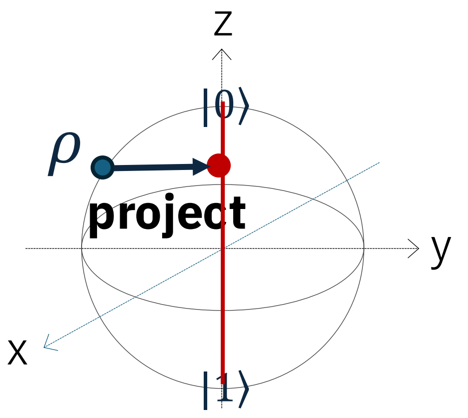
$$|\psi\rangle \mapsto |\psi_k\rangle = \frac{E_k|\psi\rangle}{\||E_k|\psi\rangle\||}$$

(draw a random number and compute one of those states)

Examples: Kraus (Operator-Sum) representation

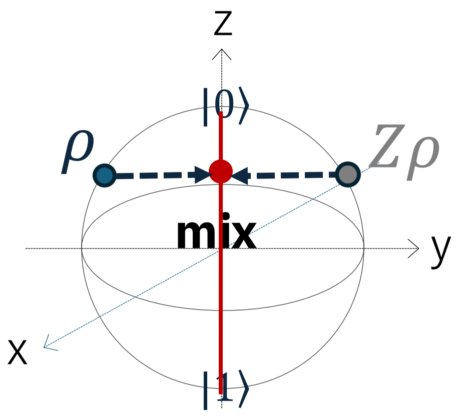
Ex. 2) Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Ex. 3) Mixture of 50% Pauli I and 50% Pauli Z

$$F_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



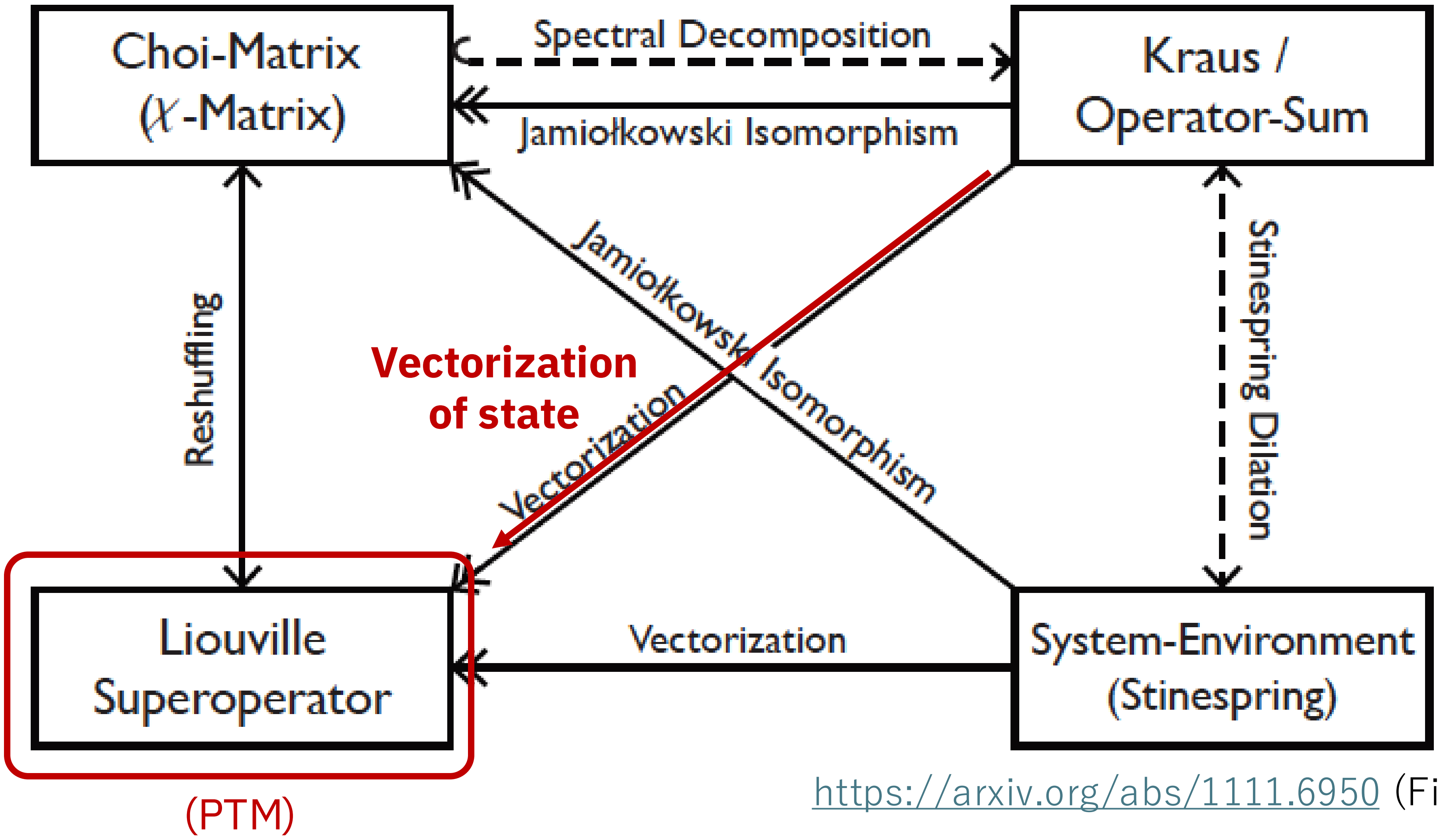
Projection to Z-axis

Those are the same channel! $E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger = F_1 \rho F_1^\dagger + F_2 \rho F_2^\dagger$

→ Kraus is not a unique representation!

Unique representation? ➔ Superoperator

Various representation of Quantum Channel



https://arxiv.org/abs/1111.6950 (Fig. 1)

See "Exploring Quantum Channels, Understanding Quantum Information & Computation: Lesson 10 by John Watrous https://www.youtube.com/watch?v=cMl-xIDSmXI" for the equivalence of Choi, Kraus and Stinespring representations)

Kraus to Superoperator transformation

Kraus (2^N x 2^N matrices)

$$\mathcal{E}^{\text{Kraus}}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{s.t.} \quad \sum_k E_k^\dagger E_k = I$$

$$\mathcal{E}^{\text{Kraus}} : \rho \mapsto \sum_k E_k \rho E_k^\dagger$$

Matrix

Vectorize state (vec trick)

Liouville **Superoperator** (4^N x 4^N matrix)

$$\mathcal{E}^{\text{SuperOp}} = \sum_k \overline{E_k} \otimes E_k = \sum_k (E_k^\dagger)^T \otimes E_k$$

$$\mathcal{E}^{\text{SuperOp}} : \text{vec}(\rho) \mapsto \left\{ \sum_k \overline{E_k} \otimes E_k \right\} \text{vec}(\rho)$$

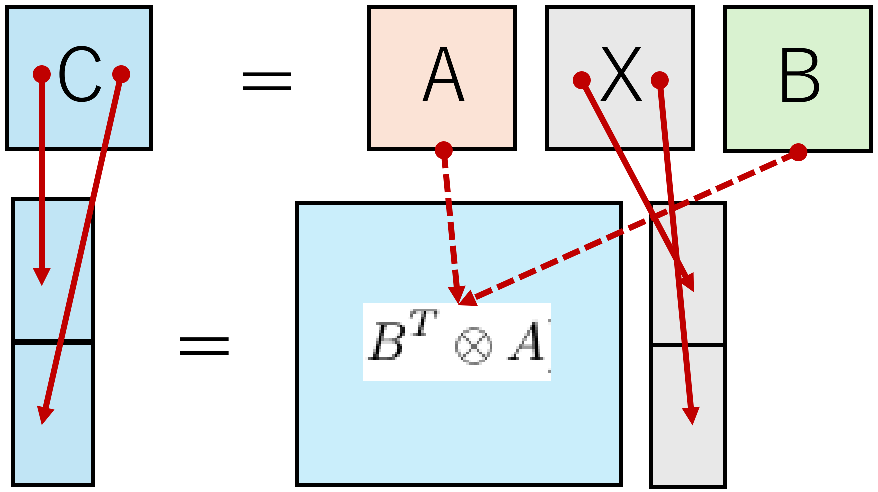
Vector

Vec trick:

$$C = A\,X\,B \Leftrightarrow \text{vec}(C) = (B^T \otimes A)\,\text{vec}(X)$$

$$\rho' = E_k\,\rho\,E_k^\dagger \Leftrightarrow \text{vec}(\rho') = ((E_k^\dagger)^T \otimes E_k)\,\text{vec}(\rho)$$

$\text{vec}(A)$ is also written as $|A\rangle\rangle$ in some literature.



Equivalence check of two quantum channels

$$\mathcal{E}^{\mathrm{SuperOp}} = \sum_k \overline{E_k} \otimes E_k$$

Ex. 2) Projection onto computational basis (0 or 1)

$$E_1 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_2 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Kraus → SuperOp

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

Ex. 3) Mixture of 50% Pauli I and 50% Pauli Z

$$E_1 = \frac{1}{\sqrt{2}} I = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E_2 = \frac{1}{\sqrt{2}} Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Kraus → SuperOp

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Superoperator is a unique representation!

9. Pauli Transfer Matrix

PTM (Pauli Transfer Matrix) is another unique representation of a quantum channel. Using PTM, you can understand that Pauli-twirling efficiently removes the off-diagonal elements.

PTM: Pauli Transfer Matrix

PTM is a superoperator with different basis

$$\mathcal{E}^{SuperOp} \xrightarrow[\text{Change of basis } (c \rightarrow \sigma)]{} \mathcal{E}^{PTM}$$

c: Computational basis

$|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|$

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| \cdots$$

σ: Pauli basis

I X Y Z

$$\rho = \frac{1}{2}(\mathrm{I} + r_x X + r_y Y + r_z Z)$$

Ex) 1-qubit basis change unitary

	00	01	10	11
I	1	0	0	1
X	0	1	1	0
Y	0	$-i$	i	0
Z	1	0	0	-1

$$T_{c\rightarrow\sigma} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

- unique channel representation
- 4^N by 4^N matrix

Ex) 1-qubit Pauli channel

SuperOp

$$\begin{bmatrix} -p_x - p_y + 1 & 0 & 0 & p_x + p_y \\ 0 & -p_x - p_y - 2p_z + 1 & p_x - p_y & 0 \\ 0 & p_x - p_y & -p_x - p_y - 2p_z + 1 & 0 \\ p_x + p_y & 0 & 0 & -p_x - p_y + 1 \end{bmatrix}$$

PTM

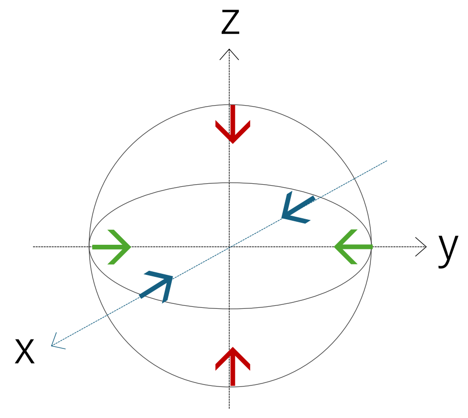
	I	X	Y	Z
I	1	0	0	0
X	0	$1-2(p_y+p_z)$	0	0
Y	0	0	$1-2(p_x+p_z)$	0
Z	0	0	0	$1-2(p_x+p_y)$

PTM of Pauli channel

PTM of Pauli channel is a **diagonal** 4^N by 4^N matrix

Ex) PTM of 1-qubit Pauli error

$$
\begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array}
\begin{array}{c}
\begin{array}{cccc} I & X & Y & Z \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1-2(p_y+p_z) & 0 & 0 \\
0 & 0 & 1-2(p_x+p_z) & 0 \\
0 & 0 & 0 & 1-2(p_x+p_y)
\end{bmatrix}
\end{array}
$$

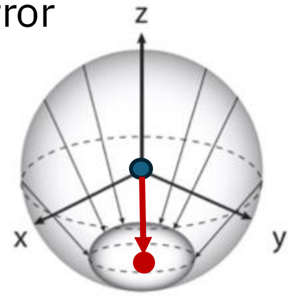


How much info on Z-axis will be kept
(1: Keep ←→0: Lost)

Where the origin shift
(No shift in Pauli channel)

Ref: PTM of Phase-amplitude damping (PAD) error

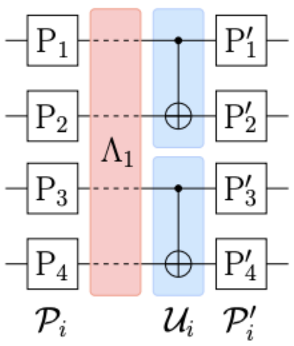
$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{1-a-b} & 0 & 0 \\
0 & 0 & \sqrt{1-a-b} & 0 \\
a(1-2p_1) & 0 & 0 & 1-a
\end{bmatrix}
$$



a: amplitude damping parameter,
b: phase damping parameter,
p_1 : excited state population (ratio)

Cited from Quantum benchmarking - Petar Jurcevic - QCHS 2022

Pauli twirling



Source: Fig 1c in [1]

Used in the first step of PEA

Pauli Twirling

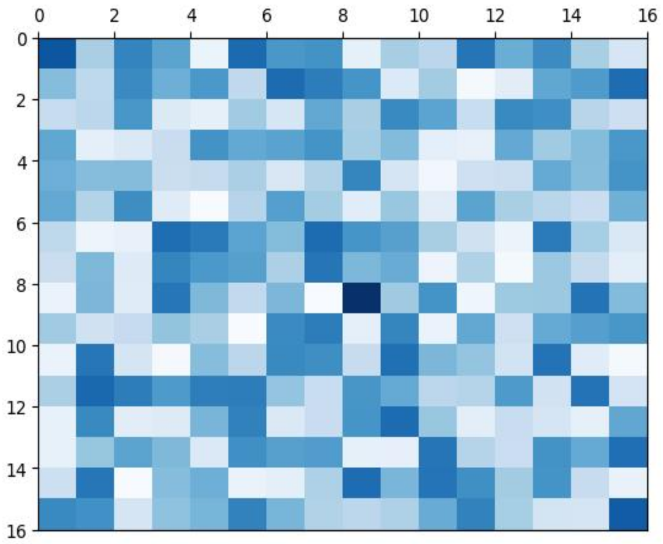
1) Simplify noise: Gate noise → Pauli channel

2) Learn noise

3) Amplify noise + ZNE

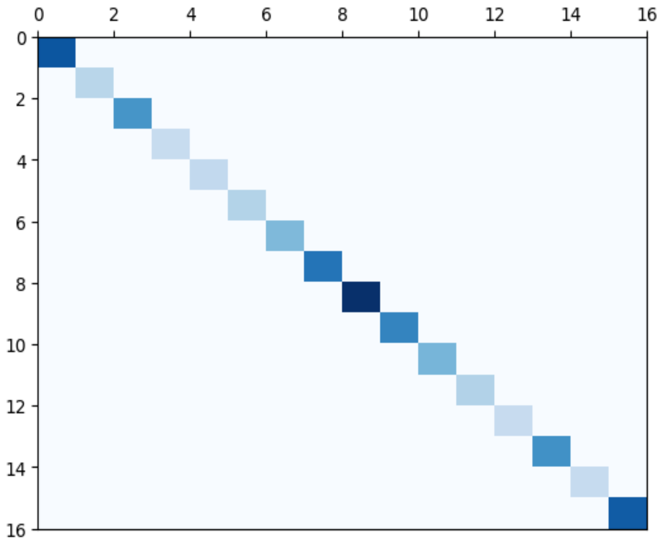
[1] Van Den Berg, E., Minev, Z. K., Kandala, A., & Temme, K. (2023). Probabilistic error cancellation with sparse Pauli–Lindblad models on noisy quantum processors. *Nature physics*, *19*(8), 1116-1121.

- Convert arbitrary error channels into Pauli channels
- PTM with off-diagonal elements → Diagonal PTM



Original gate error channel

Pauli twirling

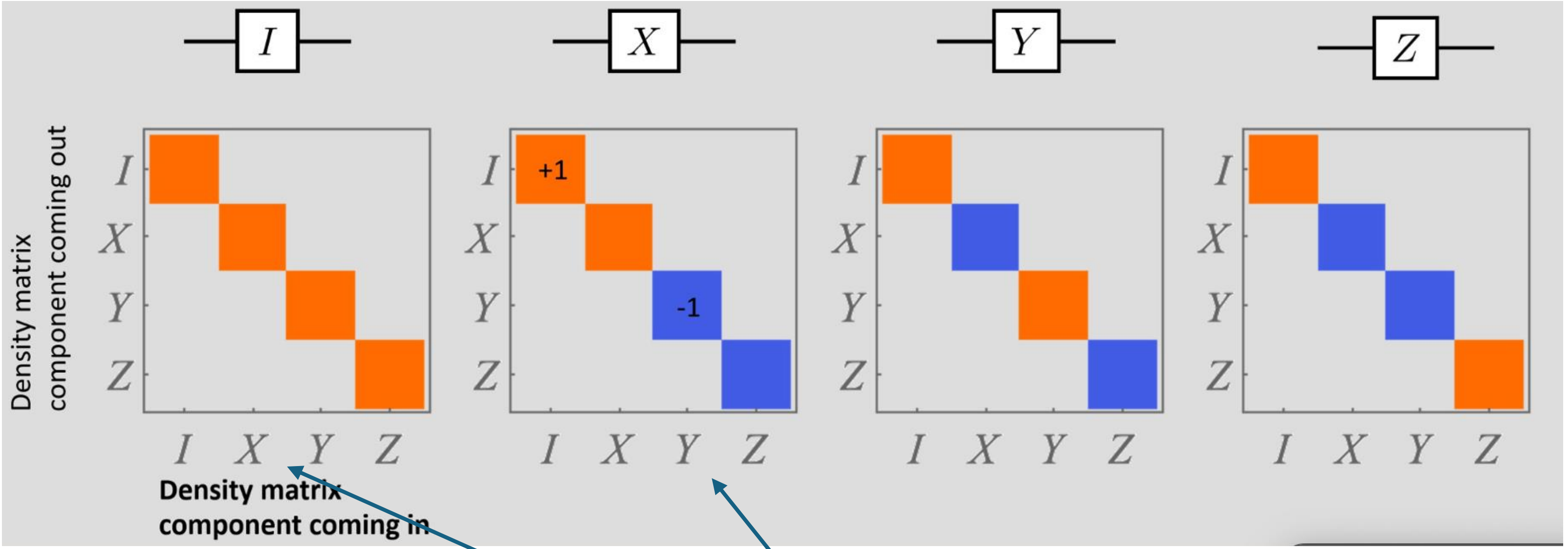


Pauli channel

Why Pauli twirling diagonalizes PTM? (1)

Zlatko Minov, A tutorial on tailoring quantum noise - Twirling 101 (https://www.zlatko-minev.com/blog/twirling)

Preparation: PTM of each Pauli gate



$p_x = p_y = p_z = 0$

$p_x = 1, p_y = p_z = 0$

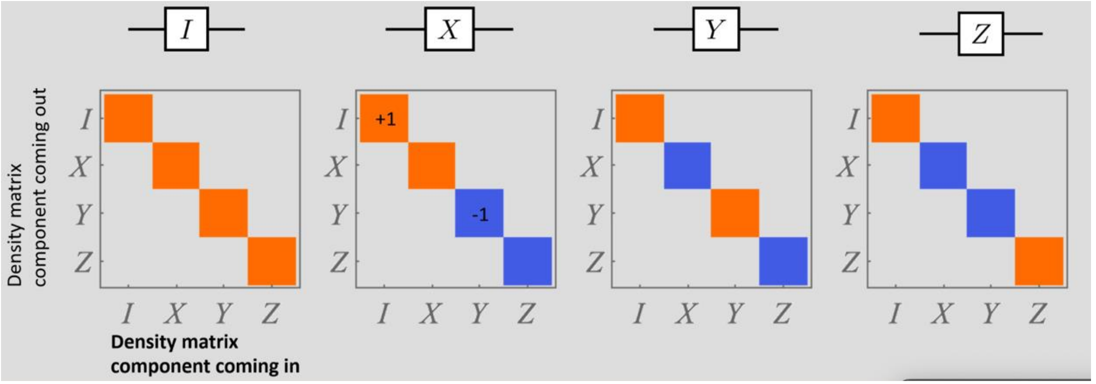
1q Pauli channel :

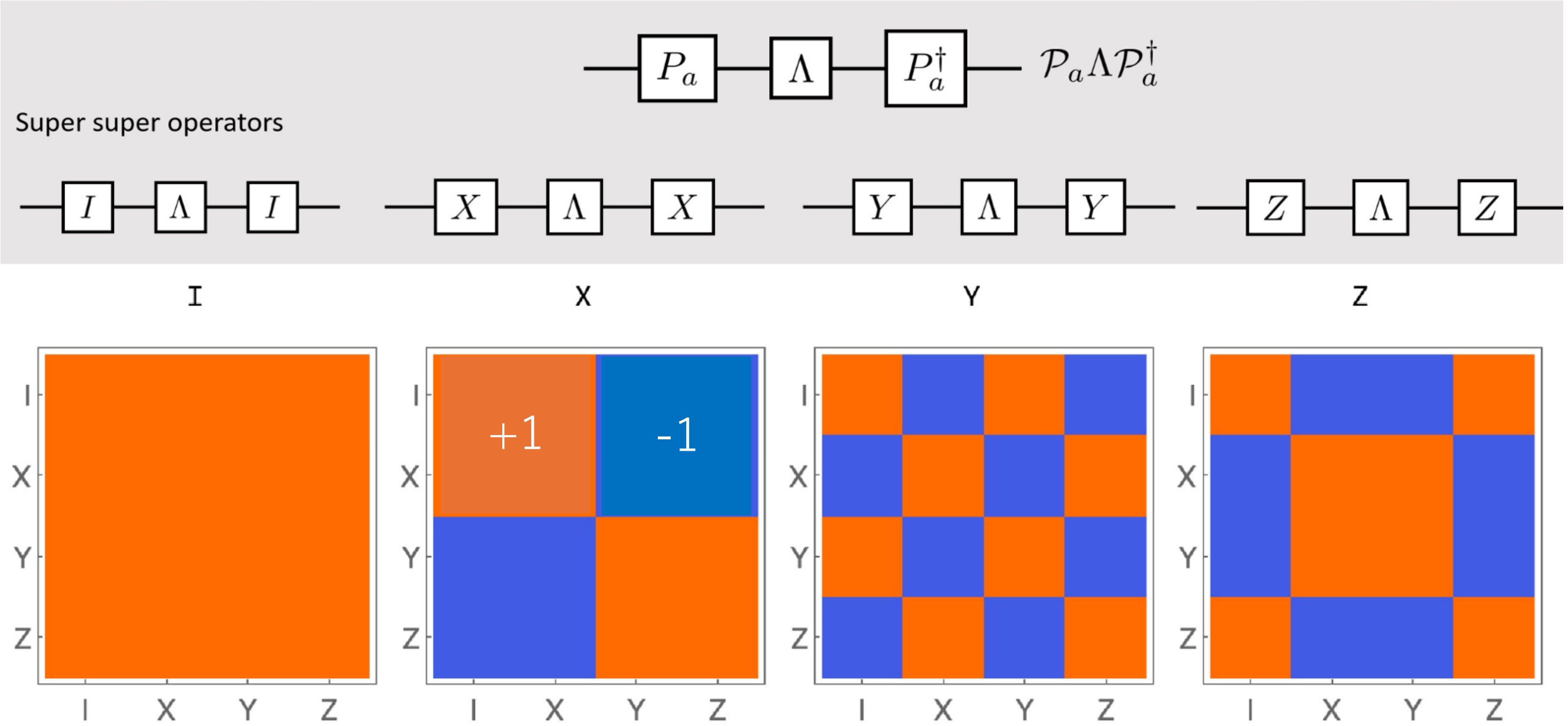
$$
\begin{array}{c|cccc}
 & I & X & Y & Z \\
\hline
I & 1 & 0 & 0 & 0 \\
X & 0 & 1-2(p_y+p_z) & 0 & 0 \\
Y & 0 & 0 & 1-2(p_x+p_z) & 0 \\
Z & 0 & 0 & 0 & 1-2(p_x+p_y)
\end{array}
$$

Why Pauli twirling diagonalizes PTM? (2)

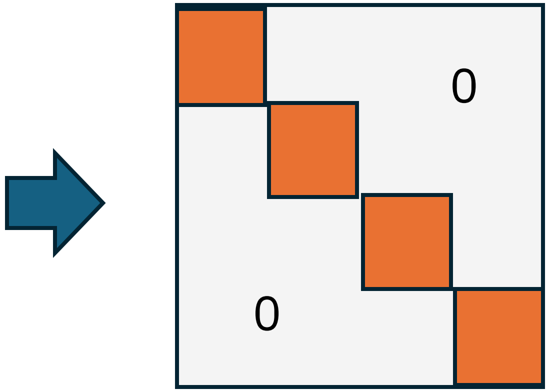
Zlatko Minov, A tutorial on tailoring quantum noise - Twirling 101
(https://www.zlatko-minev.com/blog/twirling)

PTM of each Pauli gate





Effective mask of Λ



Each Pauli pair works as a "mask" of error channel Λ (in terms of PTM)

References (Further reading)

- Introduction to Quantum Noise - Part 1 & 2 | Qiskit Global Summer School 2023
  Zlatko Minov
  - https://www.youtube.com/watch?v=3Ka11boCm1M
  - https://www.youtube.com/watch?v=gsKOx40gCUU

- Tensor networks and graphical calculus for open quantum systems
  Christopher J. Wood, Jacob D. Biamonte, David G. Cory
  - https://arxiv.org/abs/1111.6950

- A tutorial on tailoring quantum noise - Twirling 101
  Zlatko Minov
  - https://www.zlatko-minev.com/blog/twirling

- Exploring Quantum Channels | Understanding Quantum Information & Computation: Lesson 10
  John Watrous
  - https://www.youtube.com/watch?v=cMI-xIDSmXI **New** (Posted on June 6)

What you have learnt today

- What is quantum noise/error
- Error mitigation techniques
  - TREX (Twirled Readout Error eXtinction)
  - ZNE (Zero Noise Extrapolation)
  - PEA (Probabilistic Error Amplification)
- Formalism of quantum errors
  - Quantum channel
  - Standard error channels, e.g. Pauli error channel
  - PTM (Pauli Transfer Matrix) representation

Thank you